# LECTURES ON MODULI AND MIRROR SYMMETRY OF K3 SURFACES

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Abstract. This is a brief introduction to the theory of moduli and mirror families of K3 surfaces based on lectures given at the Summer Workshop on Moduli in Hamburg, August 2013.

## 1. Elliptic curves

We start with one-dimensional analogs of K3 surfaces, namely elliptic curves. Let $E$ be an elliptic curve over $\mathbb{C}$, here everything will be over $\mathbb{C}$. As a smooth 2-manifold, it is a torus $\mathbb{R}^2/\mathbb{Z}^2$. A complex structure on $E$ is defined by putting a complex structure on $\mathbb{R}^2$ and identifying $E$ with a complex 1-manifold $\mathbb{C}/\Lambda$, where $\Lambda$ is spanned by two complex numbers $\tau_1, \tau_2$, linearly independent over $\mathbb{R}$. The holomorphic form $dz$ descends to the quotient and defines a holomorphic 1-form $\omega$ on $E$ generating the space of such forms $\Omega^1(E)$. We have $H_1(E,\mathbb{Z}) \cong \mathbb{Z}^2$. Choose a basis $\gamma_1, \gamma_2$ of this group and define a vector $(z_1, z_2) = (\int_{\gamma_1}\omega, \int_{\gamma_2}\omega) \in \mathbb{C}^2$. A different choice of a basis of $\Omega^1(E)$ replaces this vector by a scalar multiple. This defines a point in $p = (z_1 : z_2) \in \mathbb{P}^1(\mathbb{C})$. The de Rham cohomology $H^1_{\mathrm{DR}}(E)$ is isomorphic to $H^1(E,\mathbb{R})$ and generated by $dx$ and $dy$, where $z = x + iy$, and hence it is also generated by $\omega = dz$ and $\bar{\omega} = d\bar{z}$. This implies that $\gamma \mapsto \int_\gamma \omega$ defines an $\mathbb{R}$-isomorphism $H^1(E,\mathbb{R}) \to \mathbb{C}$, hence $p \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}(\mathbb{R})$. This point is called the *marked period point* of $E$. Here the marking means that we have chosen an isomorphism $H_1(E,\mathbb{Z}) \to \mathbb{Z}^2$. To get rid of the marking, we see how the period changes under a change of a basis. Let $(\gamma_1' = a\gamma_1 + b\gamma_2, \gamma_2' = c\gamma_1 + d\gamma_2)$ be another basis. The matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\mathrm{GL}_2(\mathbb{Z})$. Under this change of the basis, the marked period point changes to $(az_1 + bz_2 : cz_1 + dz_2)$. The group $\mathrm{GL}_2(\mathbb{Z})$ acts on $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ by fractional-linear transformations, i.e. by automorphisms of $\mathbb{P}^1$ of the form $z \mapsto \frac{az+b}{cz+d}$. The $\mathrm{GL}_2(\mathbb{Z})$-orbit of $p$ is called the *period* of $E$. Note that $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ is equal to the union of the *upper half-plane* $\mathbb{H} = \{z = a + bi \in \mathbb{C} : b > 0\}$ and the lower half-plane $\{z = a + bi \in \mathbb{C} : b < 0\}$. The latter is equal to the image of $\mathbb{H}$ under any transformation from $A \in \mathrm{GL}_2(Z)$ with $\det(A) = -1$. These transformation represent one of the two cosets of $\mathrm{GL}_2(\mathbb{Z})$ by the normal subgroup $\mathrm{SL}_2(\mathbb{Z})$. Thus, we may assume, that the period of $E$ belongs to

$$\mathbb{P}^1(\mathbb{C}) \setminus \mathrm{GL}_2(\mathbb{Z}) = \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}).$$

In particular, one can choose a basis $(\gamma_1,\gamma_2)$ in $H_1(E,\mathbb{Z})$ such that $\int_{\gamma_1}\omega = 1, \int_{\gamma_2} = \tau \in \mathbb{H}$ and represent the period of $E$ by the orbit $\mathrm{SL}_2(\mathbb{Z})\cdot\tau$. One can reconstruct the isomorphism class of $E$ from this orbit, by taking $E = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau'$, where $\tau'\in \mathrm{SL}_2(\mathbb{Z})\cdot\tau$. The analytic structure on the orbit space is isomorphic to the complex structure of the affine line $\mathbb{A}^1$. The map $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})\to\mathbb{A}^1$ is defined by the *absolute invariant* function $j:\mathbb{H}\to\mathbb{C}$ (see [5], IV,4).

Let $f:\mathcal{E}\to S$ be a smooth family of elliptic curves over some analytic variety $S$. We assume that it is equipped with a holomorphic section, so that we can put a group structure on all fibers identifying them with complex tori. Let $R^1f_*\mathbb{Z}$ be a local coefficient system (i.e. a locally constant sheaf of abelian groups) with fibers $H^1(f^{-1}(s),\mathbb{Z})$. Locally, over some sufficiently small open set $U$, we can choose a basis of $R^1f_*\mathbb{Z}$ and $R^1f_*\Omega^1_{\mathcal{E}/S}$ to define the marked period map $U\to\mathbb{P}^1(\mathbb{C})\setminus\mathbb{P}^1(\mathbb{R})$ and the period map $U\to\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. These maps are glued together to define a holomorphic map

$$\mathrm{per}_f : S\to\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) = \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}).$$

This defines a morphism of the (analytic) stack of families of elliptic curves to the analytic variety $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})\cong\mathbb{A}^1$. It is a bijection when $S$ is a point. Thus we can view $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ as the coarse moduli space $\mathcal{M}_1$ of elliptic curves.

The moduli space $\mathcal{M}_1$ is not a fine moduli space. To construct a fine moduli space, we have to put some additional structure on an elliptic curve. For example, let us fix an isomorphism $H_1(E,\mathbb{Z}/n\mathbb{Z})\to(\mathbb{Z}/n\mathbb{Z})^2$. This isomorphism should also preserve the symplectic form on $H_1(E,\mathbb{Z}/n\mathbb{Z})$ defined by the cup-product and the standard symplectic form on $(\mathbb{Z}/n\mathbb{Z})^2$ defined by the matrix $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.[1] Then the group $\Gamma$ should be replaced with the subgroup $\Gamma(n)$ preserving this structure. In particular, $\mathrm{SL}_2(\mathbb{Z})=\Gamma(1)$. We assume that $n>2$. Then $\Gamma(n)$ is lifted isomorphically to a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of matrices $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $a\equiv b\equiv 1, c\equiv b\equiv 0 \mod n$.

Let us construct a universal family over $\mathbb{H}/\Gamma(n)$. Consider the group $\widetilde{\Gamma}(n)=\mathbb{Z}^2\rtimes\Gamma(n)$, where the semi-direct product is defined by the natural embedding of $\Gamma$ in $\mathrm{SL}_2(\mathbb{Z})$. Define the action of $\widetilde{\Gamma}(n)$ on $\mathbb{C}\times\mathbb{H}$ by the formula

$$(g;(m,n)) : (z,\tau)\mapsto (\frac{z+m\tau+n}{c\tau+d},\frac{a\tau+b}{c\tau+d}).$$

Then the projection

$$\pi:\mathcal{X}(n):=\mathbb{C}\times\mathbb{H}/\widetilde{\Gamma}(n)\to\mathbb{H}/\Gamma(n)$$

is the universal family over $\mathbb{H}/\Gamma(n)$.

If $n\le 2$, the group $\Gamma(n)$ contains $-I_2$ that acts by $(z,\tau)\mapsto(-z,\tau)$, so we see that the fibers of $\pi$ are not elliptic curves but rather their quotients by the involution $a\mapsto -a$. We get the universal family of *Kummer curves*. So,

[1] if $n=2$ the latter condition is vacuous.

we should assume here that $n \geq 3$. In fact, we could replace $\Gamma(n)$ with any subgroup $\Gamma$ of finite index of $\Gamma(1)$ not containing elements of finite order to get the universal family of elliptic curve with level $\Gamma$. If $-I_2 \notin \Gamma$, then the family is universal only over the open subset of orbits of points in $\mathbb{H}$ with non-trivial stabilizer group (called *elliptic points*). For example, we can take

$$\Gamma_1(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \mod n, a, b \equiv 1 \mod n\},\ n \geq 3.$$

The corresponding moduli space is the moduli space of pairs $(E, q)$, where $q$ is a point of order $n$ on $E$. We will later use the group

$$\Gamma_0(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \mod n\}.$$

However, this group always contains $-I_2$ so we have only the coarse moduli space of pairs $(E, \lambda)$, where $\lambda$ is a subgroup of order $n$ of $E$.

Let us see how to compactify the universal family $\mathcal{X}(\Gamma) \to \mathbb{H}/\Gamma$ to get a universal family $\overline{\mathcal{X}}(\Gamma) \to X(\Gamma)$ parameterizing *stable elliptic curves with level defined by* $\Gamma$. First we compactify the base $\mathbb{H}/\Gamma$ to get a smooth projective curve $X(\Gamma)$, called the *modular curve* of level $\Gamma$. Let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \subset \mathbb{P}^1(\mathbb{C})$. The points in $\mathbb{P}^1(\mathbb{Q})$ are called *rational boundary components.*

First we make $\mathbb{H}^*$ a topological space. We define a basis of open neighborhoods of $\infty$ as the set of open sets of the form

$$U_c = \{\tau \in \mathbb{H} : \text{Im } \tau > c\} \cup \{\infty\}, \tag{1}$$

where $c$ is a positive real number. Since $\Gamma(1)$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$, we can take for a basis of open neighborhoods of each $x \in \mathbb{H}^* \setminus \mathbb{H}$ the set of $g$-translates of the sets $U_c$ for all $c > 0$ and all $g \in \Gamma(1)$ such that $g \cdot \infty = x$. If $x \neq \infty$, each $g(U_c)$ is equal to the union of the point $x$ and the interior of the disk of some radius $r$ touching the real line at the point $x$. Now the topology on $\mathbb{H}^*/\Gamma$ is defined as the usual quotient topology: an open set in $\mathcal{H}^*/\Gamma$ is open if and only if its pre-image in $\mathcal{H}$ is open. The orbits of point in $\mathbb{H}^* \setminus \mathbb{H}$ are called *cusps.* We can choose $c$ large enough such that $\{g \in \Gamma : g(U_c) \cap U_c \neq \emptyset$ is equal to $\Gamma_\infty$. This shows that the preimage of some open neighborhood of a cusp is homeomorphic to the disjoint union of some neighborhoods of its preimage in $\mathbb{H}^*$ which we may assume to be the $\Gamma$-translates of some neighborhoods $U_c$ of $\infty$. Next we put complex structure on $U_c \cup \{\infty\}$ by considering the $\Gamma_\infty$-equivariant map $U_c \to \Delta_{e^{-2\pi c}} := \{z \in \mathbb{C} : |z| < e^{-2\pi c}\}$ given by the function $e^{2\pi i\tau/k}$, where $k$ is the index of $\Gamma_\infty$ in $\Gamma(1)_\infty/\{\pm 1\}$. This equips the orbit space $\mathbb{H}^*/\Gamma$ with a structure of a locally ringed space locally isomorphic to an open disk. The topological space $\mathbb{H}^*$ is Hausdorff, and so its quotient by $\Gamma$. Thus $\mathbb{H}^*/\Gamma$ acquires a structure of a complex manifold of dimension 1. We know that $\mathbb{H}/\Gamma(1) \cong \mathbb{A}^1$, so $\mathbb{H}^*/\Gamma(1)$ must be isomorphic to $\mathbb{P}^1$. Now $\mathbb{H}^*/\Gamma$ is a finite surjective cover of $\mathbb{H}^*/\Gamma(1)$ of complex manifolds. It must be compact too. So, we equipped $\mathbb{H}^*/\Gamma$ with a structure of a compact Riemann surface, it defines a unique structure of a projective algebraic curve on $\mathbb{H}^*/\Gamma$. This curve is denoted by $X(\Gamma)$ and is called the *modular curve* of level $\Gamma$.

Each cusp on $X(\Gamma)$ comes with its *width* or *index*, the index of $\Gamma_x$ in $\Gamma(1)_\infty/\{\pm 1\}$.

We compactify the universal family over each cusp and glue together these compactifications. Let us restrict ourselves with the cusp $\Gamma \cdot \infty$ of width $k$, other cusps are dealt similarly, by changing the coordinates in $\mathbb{P}^1(\mathbb{C})$. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{H} & \xrightarrow{\exp} & \mathbb{C}^* \times \Delta^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathbb{H}/\Gamma(n) & \xleftarrow[j]{} & \Delta^* \end{array}$$

where $\exp : (z, \tau) \mapsto (e^{2\pi i z}, e^{2\pi i \tau / k})$, $\Delta^* = \{z \in \mathbb{C} : |z| < 1\}$ and $\pi'$ is the second projection. The map $j$ is an isomorphism onto the quotient of $U_c$ by $\Gamma_\infty$ for some positive $c$ (in fact, for $c > 1$) identified with the the disk $\Delta$ of radius $e^{-2\pi c}$. Now we need to fill in $\Delta^*$ with a point, the center of the disk, and to fill in $\mathbb{C}^* \times D^*$ over this point with a stable genus one curve.

To do this we use toric geometry. Identify $\mathbb{C}^* \times \Delta^*$ with an open (analytic) subset of $\mathbb{C}^* \times \mathbb{C}^*$. Let us use a partial toric completion of $\mathbb{C}^* \times \mathbb{C}^*$ by using the fan $\Sigma$ defined by the rays $\mathbb{R}_+(m, 1)$, where $m \in \mathbb{Z}$. Each cone $\sigma_m = \mathbb{R}_+(m, 1) + \mathbb{R}_+(m + 1, 1)$ defines an affine toric variety isomorphic to the affine plane. Gluing them together defines a smooth scheme $X_\Sigma$ of locally finite type. The canonical projection $Y \to \mathbb{C}^* \times \mathbb{C}$ is a birational morphism with the exceptional divisor equal to the union of an infinite chain of $\mathbb{P}^1$'s intersecting transversally at one point. Since $-2(m, 1) + (m + 1, 1) + (m - 1, 1) = 0$, the theory of toric varieties gives us that the self-intersection of each exceptional curve is equal to $-2$. Our fan $\Sigma$ is invariant with respect to the natural action of $\Gamma_\infty$ on $\mathbb{R}^2$ via $(x, y) \mapsto (x, y + k)$. And this action coincides with the action of the group on the open torus $\mathbb{C}^* \times \mathbb{C}^*$ contained in $X_\Sigma$. Now to define our compactification we consider the quotient $X_\Sigma/\Gamma_\infty$ and restrict the projection $X_\Sigma/\Gamma_\infty \to \mathbb{C}$ over $\Delta \subset \mathbb{C}$. The fiber of this projection is a polygon of $n$ curves with self-intersection $(-2)$. We repeat this procedure over each orbit of $\Gamma_\infty$ in $\mathbb{P}^1(\mathbb{Q})$.

In this way we obtain a *modular elliptic surface* $f : S(\Gamma) \to X(\Gamma)$. It is a special case of an elliptic surface, a smooth projective surface $S$ equipped with a morphism $f : X \to C$ to a smooth projective curve such that a general fiber is a smooth elliptic curve. We assume that, as in the case of modular elliptic surface, the morphism has a section $s : C \to X$. It defines a structure of a group on each nonsingular fiber (or even on the set of smooth points of each fiber). The singular fibers have beed classified by K. Kodaira. We assume additionally that $f : X \to C$ is minimal in the sense that the map does not factor through any other elliptic surface $X' \to C$. This can be achieved by blowing down all smooth rational curves on $X$ with self-intersection $-1$ contained in fibers of $f$. Then the singular fibers could be of the following types.

We consider each fiber $X_t$ as an effective divisor $\sum_{i=}^{r} n_i R_i$ on $X$. If $X_t$ is irreducible, then $X_t = R_t$ is isomorphic to a curve of arithmetic genus one with either ordinary double points or an ordinary cuspidal double point. It is denoted by $I_1$ and $II$, respectively. If $X_t$ is reducible, then each component $R_t$ is a smooth rational curve with self-intersection equal to $-2$. We assign to $X_t$ a graph with vertices corresponding to the irreducible components of $X_t$ and the edges corresponding to the intersection points of the components taken with multiplicities. The graph is weighted by the multiplicities $n_i$ of the components. Here are the graphs and Kodaira's notations for the corresponding fibers.

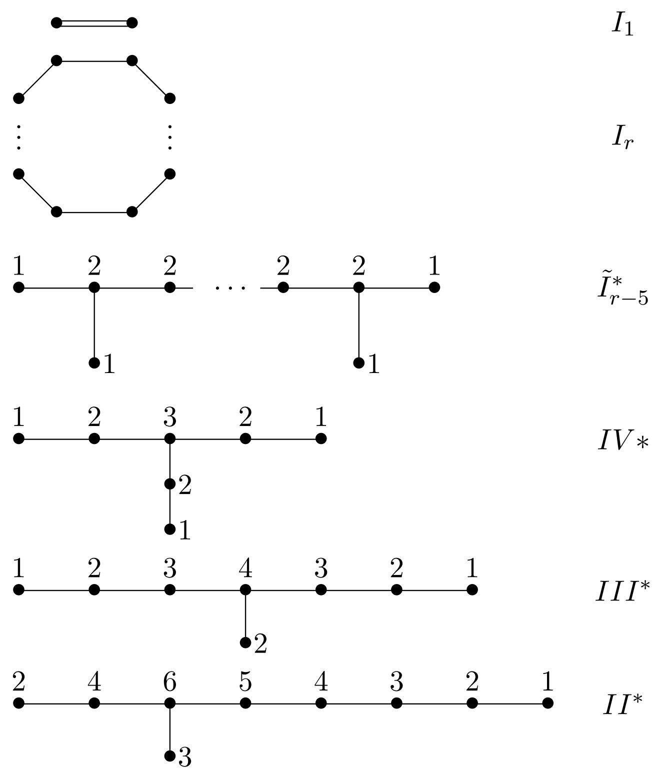


FIGURE 1. Reducible fibers of elliptic surfaces

In the case $r = 2$, there are two possibilities: either the two components intersect at two points with multiplicity 1 or are tangent at one point. Tn the first case, the fiber is of type $III$, in the second case we keep the name $I_3$. In the case $I_3$ there are also two possibilities: the three components have a common point, then fiber is of type $IV$, otherwise we keep the name $I_3$.

One recognizes the graphs as the affine Dynkin diagrams of simple Lie algebras of types $A, D, E$.

Let us consider two examples. First we assume that $\Gamma = \Gamma(n), n \geq 3$. In this case the genus of $X(n) := X(\Gamma(n))$ is equal

$$g(X(n)) = 1 + \frac{\mu_n(n-6)}{12n}.$$

The number of cusps is equal to

$$\mu_n = \tfrac{1}{2}n^3 \prod_{p|n}(1-p^2).$$

All cusps have the same width equal to $n$. We have $\mu_n$ singular fibers of $S(n) := S(\Gamma(n)) \to X(n)$ of types $I_n$. Consider any fiber $F$ , singular or not, as a CW-complex and compute the Euler-Poincaré characteristic $e(F)$. A fiber of type $I_n$ has $e(F) = 1-1+n = n$. The smooth fiber has $e(F) = 0$. Using the additivity property of Euler-Poincaré characteristic, we easily get

$$e(S(n)) = \sum_{t \in X(n)} S(n)_t = \mu_n n.$$

On the other hand, we have the Noether formula

$$c_2(S(n)) + c_1^2 = e(S(n)) = 12(1 - q + p_g),$$

where $q$ (resp. $p_g$) is the dimension of the space of holomorphic 1-forms (resp. 2-forms) on the surface $S(n)$), and $c_1 = -K_{S(n)}$ is the first Chern class of the surface. In any relatively minimal elliptic surface, $c_1^2 = 0$ because one can show that some multiple of $c_1$ is the inverse image of a divisor class on the base curve. The number $q$ is equal to the genus of $X(n)$. We get $q = p_g = 0$ if $n = 3, 4$ and $S(n)$ is a rational surface in the first case and a K3 surface in the second case. If $n = 4$, we get an elliptic K3 surface with 6 fibers of type $I_4$. If $n = 3$, the rational elliptic surface is obtained from the famous *Hesse pencil* of cubic curves

$$\lambda(x^3 + y^3 + z^3) + \mu xyz = 0.$$

We consider the rational map from $\mathbb{P}^2$ to $\mathbb{P}^1$ defined by the formula

$$(x : y : z) \mapsto (\lambda : \mu) = (-xyz : x^3 + y^3 + z^3)$$

After we resolve (minimally) its indeterminacy points by blowing up the base points of the pencil, we find a rational elliptic surface isomorphic to $S(3)$. The elliptic fibration contains 4 singular fibers of type $I_3$.

In another example, we take $\Gamma = \Gamma_1(3)$. Then $\Gamma_1(3)$ has one orbit of elliptic points with stabilizer of order 3. Over this point we have a fiber of type $IV^*$. We have two cusps $\Gamma \cdot \infty$ and $\Gamma \cdot 0$ with widths equal to 1 and 3, respectively. Adding up the Euler-Poincaré characteristics, we get $c_2 = 12$. The modular curve $X_1(3) := X(\Gamma_1(3))$ has genus 0, so we get again $p_g = 0$ and the surface $S(\Gamma_1(3))$ is rational. It can be obtained from another pencil of cubic curves

$$\lambda yz(x + y + z) + \mu x^3 = 0$$

in the same way as in the previous example.

Finally, note that the set of sections $\mathrm{MW}(X/C)$ of any elliptic surface $X \to C$ is either empty or forms an abelian group. If there is at least one singular fiber, the group is finitely generated and is called the Mordell-Weil group of the elliptic surface. In the case $X = S(n)$ it is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$. In the case $X = S(\Gamma_1(n))$ it is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

## 2. Periods of algebraic K3 surfaces

Now let us extend the previous theory to the case of complex algebraic K3 surfaces. A K3 surface $X$ is defined by the conditions $c_1(X) = -K_X = 0$ and $b_1(X) = 0$. Noether's formula

$$12(1 - q + p_g) = K_X^2 + c_2$$

implies that the second Chern class of $X$ that coincides with the Euler-Poincaré characteristic $\sum(-1)^i b_i(X)$ is equal to 24, hence $b_2(X) = 22$. As we will see later all K3 surfaces are diffeomorphic, taking $X$ to be a smooth quartic hypersurface in $\mathbb{P}^3$, we obtain that all K3 surfaces are simply-connected. By Poincarés Duality, the symmetric bilinear pairing defined by the cup-product $H_2(X,\mathbb{Z}) \times H_2(X,\mathbb{Z}) \to H_4(X,\mathbb{Z}) = \mathbb{Z}$ defines an isomorphism

$$H_2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z}) = H_2(X,\mathbb{Z})^\vee.$$

We will often identify these two groups by this isomorphism. In particular, the cup-product corresponds to the cap-product on the cohomology.

For any abelian group $A$ and a field $K$, we write $A_K = A \otimes_{\mathbb{Z}} K$. We denote the values of the cap-product on a pair $(x, y)$ from $H^2(X,\mathbb{Z})$ by $x \cdot y$ and write $x^2 := x \cdot x$. The quadratic form $x \mapsto x^2$ equips $H^2(X,\mathbb{Z})$ with a structure of a *quadratic lattice* (= free abelian group of finite rank equipped with a integral valued quadratic form). By Poincaré's Duality, the quadratic form is unimodular, i.e. it is defined by a symmetric matrix with determinant $\pm 1$.

Using the de Rham theorem and decomposing real harmonic forms in forms of type $adz_1 \wedge dz_2$ (type $(2,0)$), $adz_1 \wedge d\bar{z}$ (type $(1,1)$) and $ad\bar{z}_1 \wedge d\bar{z}_2$ (type $(0,2)$), one obtains the Hodge decomposition

$$H^2(X,\mathbb{C}) = H^2(X,\mathbb{Z})_{\mathbb{C}} = H^{20}(X) \oplus H^{11}(X) \oplus H^{02}(X) = \mathbb{C} \oplus \mathbb{C}^{20} \oplus \mathbb{C}$$

This decomposition is an orthogonal decomposition with respect to the cap-product. Let $\omega$ be a holomorphic 2-form on $X$ generating $H^{20}(X)$. Consider the plane $P$ in $H^2(X,\mathbb{R})$ spanned by $\mathrm{Re}(\omega) = \omega + \bar{\omega}$ and $\mathrm{Im}(\omega) = -i(\omega - \bar{\omega})$. We have

$$(\omega \pm \bar{\omega}) \wedge (\omega \pm \bar{\omega}) = 2\omega \wedge \bar{\omega} > 0, \ (\omega + \bar{\omega}) \wedge -i(\omega - \bar{\omega}) = 0.$$

Thus the restriction of the cap-product to $P$ is positive definite, and also $P$ comes equipped with a basis $(\mathrm{Re}(\omega), \mathrm{Im}(\omega))$ defining an orientation in the plane. The plane $P$ depends only on the line $\mathbb{C}\omega$ generated by $\omega$, hence $[\omega] \in \mathbb{P}(H^2(X,\mathbb{C}))$ defines a positive definite oriented plane in $H^2(X,\mathbb{R})$. Let $h \in H^2(X,\mathbb{R})$ be the class of a Kähler form on $X$ or the Chern class of an ample line bundle on $X$ in $H^2(X,\mathbb{Z})$. Then $h$ is of type $(1,1)$, $h^2>0$ and $h$ is orthogonal to $\omega,\bar{\omega}$, hence to $P$. One can show that the orthogonal complement of $h$ in $H^{11}(X)\cap H^2(X,\mathbb{R})$ is negative definite. Thus the cap-product on $H^2(X,\mathbb{R})$ is of signature $(3,19)$. Next we use that the quadratic from on $H^2(X,\mathbb{Z})$ defined by the cap-product is even, i.e. takes only even integers as its values. This follows from the *Wu formula* $x^2\equiv K_X\cdot x \mod 2$. By a theorem of J. Milnor, an even unimodular indefinite quadratic form is an orthogonal direct sum of $k$ copies of the integral hyperbolic plane $U$ defined by the matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $m_1$ copies of the quadratic form $E_8$ defined by the negative of the Cartan matrix of the simple root system of type $E_8$, and $m_2$ copies of the quadratic form of $E_8$ multiplied by $-1$. Its signature is equal to $(k+8m_2,k+8m_1)$. In our case we must have $(3,19)=(3k+8m_2,k+8m_1)$, hence $k=3,m_1=2,m_2=0$. Thus we get

$$H^2(X,\mathbb{Z})\cong U^{\oplus 3}\oplus E_8^{\oplus 2},$$

where the direct sum means the orthogonal direct sum. We denote the right-hand side lattice by $\mathbb{L}_{K3}$ and call it the *K3-lattice*.

Next we define the *marked period* of $X$ in the same manner as for an elliptic curve. Choose a basis $(\gamma_1,\ldots,\gamma_{22})$ of $H_2(X,\mathbb{Z})\cong H^2(X,\mathbb{Z})$ to define an isomorphism of lattices

$$\phi: H^2(X,\mathbb{Z})\cong\mathbb{L}_{K3},$$

called a *marking* of $X$. Then the image of $\omega$ under the isomorphism $\phi_{\mathbb{C}}: H^2(X,\mathbb{C})\to(\mathbb{L}_{K3})_{\mathbb{C}}$ can identified with the vector

$$\phi_{\mathbb{C}}(\omega)=(\int_{\gamma_1}\omega,\ldots,\int_{\gamma_{22}}\omega)\in\mathbb{C}^{22}.$$

To get rid of a choice of a generator of the one-dimensional space $H^{20}(X)$, we should consider the corresponding point $[\phi_{\mathbb{C}}(\omega)]\in\mathbb{P}((\mathbb{L}_{K3})_{\mathbb{C}})\cong\mathbb{P}^{21}$. It is called the *marked period* of $X$. Since $\omega\wedge\omega=0$, the point $[\phi(\omega)]$ belongs to the quadric hypersurface $Q$ in $\mathbb{P}((\mathbb{L}_{K3})_{\mathbb{C}})$ defined by the quadratic form of the lattice $H^2(X,\mathbb{Z})$. Also $\omega\wedge\bar{\omega}$ can be taken as a volume form on $X$, hence $\omega\wedge\bar{\omega}>0$ that shows that $[\phi(\omega)]$ belongs to an open subset $\mathcal{D}$ of $Q$ defined by the inequality $x\cdot\bar{x}>0$.

There are further restrictions on the point $[\phi(\omega)]$. Let $H_2(X,\mathbb{Z})_{\mathrm{alg}}$ be the subgroup of algebraic 2-cycles spanned by the fundamental 2-cycles of analytic (=algebraic) irreducible curves on $X$. By duality, it corresponds to a subgroup $H^2(X,\mathbb{Z})_{\mathrm{alg}}$ of $H^2(X,\mathbb{Z})$. The Chern class homomorphism $c_1:\mathrm{Pic}(X)\to H^2(X,\mathbb{Z})$ defines an isomorphism

$$c_1:\mathrm{Pic}(X)\to H^2(X,\mathbb{Z})_{\mathrm{alg}}.$$

We will denote by $S_X$ its image and call it the *Picard lattice* of $X$. By definition of the Chern class of a line bundle, the image of $c_1$ belongs to

$H^{11}(X) \cap H^2(X, \mathbb{Z})$, hence it is orthogonal to $H^{20}$, and hence to $[\phi_{\mathbb{C}}(\omega)]$.[2] Let $T_X$ denote the orthogonal complement of $S_X$ in $H^2(X, \mathbb{Z})$ and $T = \phi(T_X) \subset \mathbb{L}_{K3}$. Then

$$[\phi(\omega)] \in \mathbb{P}(T_{\mathbb{C}}) \subset \mathbb{P}((\mathbb{L}_{K3})_{\mathbb{C}}).$$

We restrict the quadratic form $Q$ to the linear subspace of $\mathbb{P}((\mathbb{L}_{K3})_{\mathbb{C}})$ defined by $T_{\mathbb{C}}$ and obtain, finally, that

$$[\phi(\omega)] \in \mathcal{D}_T \subset \mathbb{P}(T_{\mathbb{C}}) \cong \mathbb{P}^{21-\rho}.$$

Here $\mathcal{D}_T = \mathcal{D} \cap \mathbb{P}(T_{\mathbb{C}})$. By Hodge's Index Theorem, the quadratic form on $(S_X)_{\mathbb{R}}$ has signature $(1, \rho - 1)$, where $\rho = \text{rank } S_X$. Thus the signature of $T$ is equal to $(2, 19 - \rho)$.

Let $T$ be any quadratic lattice of signature $(2, n)$. Let $G^+(2, T_{\mathbb{R}})$ be the (real) Grassmann variety of positive definite oriented planes in $T_{\mathbb{R}}$. The orthogonal group $\mathrm{O}(T_{\mathbb{R}}) \cong \mathrm{O}(2, 19-\rho)$ acts transitively on this space and the stabilizer subgroup of a plane $P \in G^+(2, T_{\mathbb{R}})$ is equal to $\mathrm{SO}(P) \times \mathrm{O}(P^\perp) \cong \mathrm{SO}(2) \times \mathrm{O}(19-\rho)$. The change of the orientation decomposes $G^+(2, T_{\mathbb{R}})$ into two connected components. If we fix a connected component $G^+(2, T_{\mathbb{R}})_0$, we obtain a smooth connected homogeneous manifold[3]

$$G^+(2, T_{\mathbb{R}})_0 \cong \mathrm{SO}_0(2, 19 - \rho)/\mathrm{SO}(2) \times \mathrm{SO}(19 - \rho).$$

Consider a map $G^+(2, T_{\mathbb{R}}) \to \mathcal{D}_T$ defined by assigning to a plane $P$ spanned by an orthogonal oriented basis $v, w$ the complex line in $T_{\mathbb{C}}$ generated by $v + iw$. We have $(v+iw)^2 = v^2 - w^2 = 0$ and $(v+iw)(v-iw) = v^2 + w^2 > 0$. Thus, the image of the map belongs to $\mathcal{D}_T$. It is easy to see that this defines a diffeomorphism of smooth manifolds $G^+(2, T_{\mathbb{R}}) \to \mathcal{D}_T$, and by transfer of the complex structure of $Q_T$, we equip $G^+(2, T_{\mathbb{R}})$ with a structure of a complex homogeneous space. The two connected components are permuted by the conjugation involution.

Each connected component of $\mathcal{D}_T$ is a Hermitian symmetric space of orthogonal type (or of Cartan's Type IV).[4] The special orthogonal group $\mathrm{SO}(T)$ acts properly discontinuously on this space. The theory of automorphic forms on Hermitian homogeneous spaces shows that the orbit space $\mathcal{D}_T$ has a natural structure of a quasi-projective algebraic variety.

So far, we have defined the marked period of a K3-surface. To get rid of the marking, we have to see how the period point changes under a change of a basis of $H^2(X, \mathbb{Z})$. A change of a basis corresponds to an action of the group

[2] Another way to see it is to use that a local coordinate $z$ on an open subset of an irreducible algebraic curve is a part of local coordinates $z, z'$ on the surface. The 2-form $\omega$ can be locally given as $a(z, z')dz \wedge dz'$, hence integrating over the curve we get zero. The converse is called the Lefschetz Theorem:if $\int_\gamma \omega = 0$, then $\gamma$ is an algebraic cycle.

[3] The real Lie group $\mathrm{O}(T_{\mathbb{R}}) \cong \mathrm{O}(2, n)$ has four connected components, the group $\mathrm{SO}(2, n)$ consists of two connected components, the connected component of the identity $SO_0(2, n)$ is equal to the kernel of the spinor norm (see, for example, [4]).

[4] Another example of a Hermitian symmetric space is the Siegel half-planes $\mathcal{H}_g$. It is of type III, in Cartan's classification.

$\mathrm{O}(\mathbb{L}_{K3})$ of isometries of the K3-lattice. Let $\mathrm{O}(\mathbb{L}_{K3})'$ be the subgroup of this group that consists of isometries preserving the orthogonal decomposition $S \oplus T := \phi(S_X) \oplus \phi(T_X)$. There is a natural projections

$$\alpha : \mathrm{O}(\mathbb{L}_{K3})' \to \mathrm{O}(S), \quad \beta : \mathrm{O}(\mathbb{L}_{K3})' \to \mathrm{O}(T).$$

Let

$$\mathrm{O}(T)^* = \beta(\mathrm{Ker}(\alpha)) = \{\sigma \in \mathrm{O}(T) : \exists \tilde{\sigma} \in \mathrm{O}(\mathbb{L}_{K3})' : \beta(\tilde{\sigma}) = \sigma, \alpha(\tilde{\sigma}) = \mathrm{id}_S\}.$$

For any even lattice $L$ we have a canonical map $L \to L^\vee$ defined by the symmetric bilinear form associated to the quadratic form of the lattice. If the quadratic form is non-degenerate, the quotient group $A_L = L^\vee/L$ is a finite abelian group of order equal to the absolute value of the determinant of any symmetric matrix representing the quadratic form of the lattice. We equip $A_L$ with a quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$ by extending the quadratic form of $L$ to $L^\vee \subset L_\mathbb{Q}$ and setting

$$q_{A_L}(x + L) = \tfrac{1}{2}x^2 \mod 2\mathbb{Z}.$$

The pair $(A_L, q_{A_L})$ is called the *discriminant quadratic group* of $L$. We have a canonical homomorphism $\mathrm{O}(L) \to \mathrm{O}(A_L)$ and we define $\mathrm{O}(L)^*$ to be the kernel of this homomorphism. If $L$ is embedded in a unimodular lattice $N$ with torsion-free quotient (this is called a *primitive embedding*), with orthogonal complement $L^\perp$, then $(A_L, q_{A_L}) \cong (A_L, -q_{A_{L^\perp}})$ and

$$\mathrm{O}(L)^* = \{\sigma \in \mathrm{O}(T) : \exists \bar{\sigma} \in \mathrm{O}(N) \text{ such that } \bar{\sigma}|_L = \sigma, \bar{\sigma}|_{L^\perp} = \mathrm{id}_{L^\perp}\}.$$

Applying this to our situation we see an equivalent definition of our group $\mathrm{O}(T)^*$.

Now we can define an *unmarked period* of $X$ by taking the image of $[\phi_\mathbb{C}(\omega)]$ in $\mathcal{D}_T/\mathrm{O}(T)^*$.

Let $X$ be a K3 surface and let $\mathrm{Nef}(X)_\mathbb{R}$ (resp. $\mathrm{Nef}^+(X)$) be its nef (resp. ample) cone generated in $(S_X)_\mathbb{R}$ by nef (resp. ample) divisors classes. Recall that a divisor on a nonsingular projective surface is called *nef* if its intersection with any curve on the surface is non-negative. It $D \cdot C > 0$ for all curves and $D^2 > 0$, then it is also ample. If $D^2 \geq 0$ and $D \cdot C < 0$, we get $C^2 < 0$. This follows from the fact that the signature of the Picard lattice of any smooth surface is equal to $(1, \rho - 1)$. In our case, by the adjunction formula $C^2 + C \cdot K = -2\chi(\mathcal{O}_C)$, we get $C^2 = -2, C \cong \mathbb{P}^1$. So, if $X$ has no smooth rational curves, all effective divisors with positive self-intersection are ample. One may express this in a little more sophisticated way. Let $W_X$ be the subgroup of $\mathrm{O}(S_X)$ generated by the isometries of the form $r_\delta : x \mapsto x + (x \cdot \delta)[\delta$, where $\delta$ is the divisor class of a smooth rational curve on $X$. Choose the connected component $(S_X)^+_\mathbb{R}$ of the cone $\{x \in (S_X)_\mathbb{R} : x^2 > 0\}$ that contains an ample divisor class. For any $C$ the hypersurfaces $\delta^\perp$ in $(S_X)^+_\mathbb{R}$ are the mirrors of these reflections, i.e. the sets of fixed points. The complement of the union of mirrors is the union of connected components permuted by $W_X$. In fact, each of them can be taken as a fundamental domain for the action of $W_X$ in $(S_X)^+_{\mathbb{R}}$. The ample cone $\mathrm{Nef}^+(X)$ is one of them and its closure is the nef cone.

Next, we have to do everything for families.

We fix a primitive embedding $M \hookrightarrow \mathbb{L}_{K3}$ of a lattice $M$ of signature $(1, \rho-1)$. We will identify $M$ with its image in $\mathbb{L}_{K3}$. The set $\{x \in M_{\mathbb{R}} : x^2 > 0\}$ consists of two connected components. In orthogonal coordinates $(x_1, \ldots, x_\rho)$ they differ by the sign of $x_1$. Fix one of its connected components and denote it by $M^+_{\mathbb{R}}$. Let

$$\Delta_M = \{\delta \in M : \delta^2 = -2\}$$

and $W_M$ be the 2-*reflection group* of $M$, the subgroup of $\mathrm{O}(M)$ generated by isometries

$$s_\delta : x \mapsto x + (x \cdot \delta)\delta, \quad \delta \in \Delta(M).$$

We choose a fundamental domain $\Pi_M$ for the action of $W_M$ in $M^+_{\mathbb{R}}$. It is a convex polyhedral cone bounded by intersections of hyperplanes $\delta^\perp$ with $M^+_{\mathbb{R}}$.

We define a *$M$-polarization* of $X$ to be a lattice embedding $\iota : M \hookrightarrow S_X$ such that $j(\Pi_M) \cap \mathrm{Nef}(X) \neq \emptyset$. An $M$-polarization is called *ample* if $j(\Pi_M) \cap \mathrm{Nef}^+(X) \neq \emptyset$. A *marking of a $M$-polarized surface* $X$ is a marking $\phi : H^2(X, \mathbb{Z}) \to \mathbb{L}_{K3}$ such that the composition $\phi \circ j : M \to \mathbb{L}_{K3}$ coincides with $\iota$. Note that $\mathbb{L}_{K3} \cong \iota(M) \oplus \iota(M)^\perp$, so any $M$-polarization $j : M \to S_X \subset H^2(X, \mathbb{Z})$ can be extended to a marking $\phi : H^2(X, \mathbb{Z}) \to \mathbb{L}_{K3}$.

A smooth family $f : \mathcal{X} \to \mathsf{S}$ of K3 surfaces defines a local coefficient system $R^2 f_* \mathbb{Z}$ on $S$ with fibers $H^2(\mathcal{X}_s, \mathbb{Z})$. A *$M$-polarization* of the family is an injection of the constant local coefficient system $j : M_S \hookrightarrow R^2 f_* \mathbb{Z}$ such that the maps of fibers $j_s : M \to H^2(\mathcal{X}_s, \mathbb{Z})$ defines a $M$-marking $j_s : M \to S_{\mathcal{X}_s} \subset H^2(\mathcal{X}_s, \mathbb{Z})$. A marking of the family of $M$-polarized surfaces is an isomorphism of local coefficient systems $\phi : R^2 f_* \mathbb{Z} \to (\mathbb{L}_{K3})_S$ such that $j_s \circ \phi_s : M \to \mathbb{L}_{K3}$ coincides with $\iota$. Let $N = M^\perp$ (in $\mathbb{L}_{K3}$) and $\mathcal{D}_N$ be the corresponding period domain. We define the *period map* of a marked $M$-polarized family $f : \mathcal{X} \to \mathsf{S}$

$$\mathrm{per}_f : \mathsf{S} \to \mathcal{D}_N, \quad s \mapsto [\phi_s(\omega_s)] \in \mathcal{D}_N \subset \mathbb{P}(N_{\mathbb{C}}),$$

where $\omega_s$ is a generator of $H^{20}(\mathcal{X}_s)$.

If $f$ is a family of not necessary marked $M$-polarized K3 surfaces, we consider the universal cover $\tilde{S}$ and the corresponding base-change family $\tilde{f} : \widetilde{\mathcal{X}} \to \tilde{\mathsf{S}}$. We fix a trivialization of the local coefficient system $R^2 \tilde{f}_* \mathbb{Z}$ to define a marked family. then we define the *period map*

$$\mathrm{per}_f : \mathsf{S} \to \mathcal{D}_N / \mathrm{O}(N)^*$$

as the composition of $\mathrm{per}_{\tilde{f}} : \tilde{S} \to \mathcal{D}$ and the quotient map $\mathcal{D}_N \to \mathcal{D}_N / \mathrm{O}(N)^*$. One can show that both marked and unmarked period maps are holomorphic maps.

For any $\delta \in N$ with $\delta^2 = -2$, let

$$H_\delta = \mathbb{P}((\delta^\perp)_\mathbb{C}) \cap \mathcal{D}_T.$$

Suppose the period point of a marked $M$-polarized K3 surface belongs to $H_\delta$. Let $\phi(\gamma) = \delta$ for some $\gamma \in j(M)^\perp$. This means that $\int_\gamma \omega = 0$, hence, by Lefschetz's Theorem, $\gamma \in H^2_{\text{alg}}(X, \mathbb{Z})$. Since every element in $j(M)$ is orthogonal to $\gamma$, it cannot be an ample divisor in $X$. So, we will never get a point in $H_\delta$ as the period point of an ample $M$-polarized K3-surface $X$. The surface $X$ will contain some nef divisors but not ample ones. So, if we have a family $\mathcal{X} \to \mathsf{S}$ of $M$-polarized surfaces, we will not be able to embed simultaneously all members of the family in a projective space by means of a divisor class coming $h$ from $j(M)$. If choose any $h \in C(M)^+$, then it will define a nef divisor $h_s$ on each $\mathcal{X}_s$. For some members $\mathcal{X}_s$ the divisor class $h_s$ will not be ample. So, the linear system $|h|$ will map $\mathcal{X}_s$ into projective space and the image will be a singular surface with rational double points, the images of smooth rational curves $C$ such that $h \cdot C = 0$. The classes of these curves $C$ on $\mathcal{X}_s$ such that $h_s \cdot C = 0$ will not belong to $j_s(M)$. In particular, the rank of the Picard lattice of $\mathcal{X}_s$ must be greater than the rank of $M$.

The following theorem was proved by I.R. Shafarevich and I. I. Piatetsky-Shapiro in the seventies. It goes under the name *Global Torelli Theorem.* In my opinion, it is one of the deepest results in mathematics.

**Theorem 1.** *The quasi-projective variety*

$$\mathcal{M}_{K3,M} := \mathcal{D}_N/\mathrm{O}(N)^*, \quad (\textit{resp. } \mathcal{M}^a_{K3,M} = (\mathcal{D}_N \setminus \cup_\delta H_\delta)/\mathrm{O}(N)^*)$$

*is the coarse moduli space for families of $M$-polarized (resp. ample $M$-polarized K3-surfaces).*

In plain words it means that one can reconstruct the isomorphism class of a $M$-polarized K3-surface by the vector $(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_{22}} \omega)$.

## 3. Compactification of the moduli space

The homogeneous space $\mathcal{D}_N$ admits a partial compactification by adding *rational boundary components* to $\mathcal{D}_N$, similar to the case of the upper half-plane. A boundary component is a maximal connected complex analytic submanifold of the boundary of a connected component of $\mathcal{D}_N$ in its closure in the quadric $Q_N$. A boundary component is called rational if its stabilizer subgroup in $\mathrm{O}(N_\mathbb{R})$ can be defined over $\mathbb{Q}$, i.e. it preserves some lattice in $N_\mathbb{Q}$. Rational boundary components correspond to isotropic subspaces in $N_Q$, or, equivalently, primitive isotropic sublattices of $N$. We use the Grassmannian model $G^+(2, N_\mathbb{R})$ of $\mathcal{D}_N$. The boundary of $G^+(2, N_\mathbb{R})$ consists of semi-definite oriented planes. If $I$ is a one-dimensional subspace, then the set of semi-definite planes with one-dimensional radical equal to $I$ belongs to the boundary. It is equal to the cone $\mathcal{C}_I$ of vectors in $(I/I^2)^+_\mathbb{R}$ of positive norm. If $J$ is a two-dimensional isotropic subspace, then a choice of a connected component $G^+(2,N_\mathbb{R})_0$ of $G^+(2,N_\mathbb{R})$ chooses a connected component $C_J$ in $\Lambda^2 J\setminus\{0\}$. The half-plane $\Lambda^2 J + iC_J\subset\Lambda^2 J_\mathbb{C}$ lies in the boundary of $\mathcal{C}_I$ if $I\subset J$ and, when $J$ is defined over $\mathbb{Q}$, defines a rational boundary component.[5]

For each isotropic line $I\subset N_\mathbb{R}$ one defines a *tube domain realization* of $\mathcal{D}_N$ by taking the image of $\mathcal{D}_N$ under the projection $\Pi_I:\mathbb{P}(N_\mathbb{C})\dashrightarrow\mathbb{P}(N_\mathbb{C}/I_\mathbb{C})$. Recall that, under the projection of a quadric $Q$ from its point $x_0\in Q$, the image of $Q\setminus\{x_0\}$ is contained in the complement of a hyperplane $H$ equal to the projection of the embedded tangent hyperplane $T_{x_0}Q$ of $Q$ at $x_0$. The projection blows down $T_{x_0}Q\cap Q$ to a quadric in $H$. Since a point in $\mathcal{D}_N$ defines a positive definite plane in $N_\mathbb{R}$, it cannot be orthogonal to $I$, hence it does not belong to $T_{x_0}$. This shows that $\mathcal{D}_N$ is projected isomorphically into the affine space

$$A_f=\{z=x+iy\in N_\mathbb{C}, z\cdot f=1\}/I_\mathbb{C}\subset\mathbb{P}((N/I)_\mathbb{C})\setminus\mathbb{P}((I^\perp/I)_\mathbb{C})\cong\mathbb{C}^{20-\rho}$$

where $f$ generates $I$. The condition that $[z]\in\mathcal{D}_N$ can be expressed by $x^2-y^2=x\cdot y=0, x^2+y^2>0$, this gives $y^2>0$. Restricting this isomorphism to a connected component of $\mathcal{D}_N^o$, we obtain that $\mathcal{D}_N^o$ becomes isomorphic to a *tube domain*

$$\mathcal{D}_N^o\cong\pi_I(\mathcal{D}_N^o)=V_f+iC_{I^\perp/I},$$

where $V_f=\{x\in N_\mathbb{R}: x\cdot f=1\}/I$ and $C_{I^\perp/I}$ is a connected component of $\{y\in(I^\perp/I)_\mathbb{R}: y^2>0\}$.[6] If we choose an isotropic vector $g\in N_\mathbb{R}$ such that $f\cdot g=1$, then the map $x\mapsto x-g$ will identify $V_f$ with the orthogonal complement of the hyperbolic plane $U_\mathbb{R}$ spanned by $f$ and $g$. Also, $(I^\perp/I)_\mathbb{R}$ is naturally identified with $U_\mathbb{R}^\perp$. Thus

$$\pi_I(\mathcal{D}_N)=U_\mathbb{R}^\perp+iC_{U^\perp}.$$

Note that $\mathrm{id}_{U_\mathbb{R}}\oplus-\mathrm{id}_{U_\mathbb{R}^\perp}$ switches the two connected components $\pi_I(\mathcal{D}_N)$. The explicit isomorphism $\pi_I(\mathcal{D}_N)\to\mathcal{D}_N$ is defined by the formula $z\mapsto[z+g-\frac{1}{2}z^2f]$

The hyperbolic plane in $N_\mathbb{R}$ generated by $f$ and $g$ may be not defined over $\mathbb{Z}$. We say that a primitive isotropic vector $f$ in $N$ is $m$-*admissible* if there exists an isotropic vector $g$ with $f\cdot g=m>0$ and $m$ generates the image of the map $N\to\mathbb{Z}, x\mapsto x\cdot f$. One can show that in this case the pair $f,g$ generates a sublattice $\mathbb{Z}f+\mathbb{Z}g$ isomorphic to the lattice $U(m)$ obtained from $U$ by multiplying its quadratic form by $m$, and $N\cong(\mathbb{Z}f+\mathbb{Z}g)\oplus(\mathbb{Z}f+\mathbb{Z}g)^\perp$. It follows from this that $I^\perp/I\cong(\mathbb{Z}f+\mathbb{Z}g)^\perp$, and, in particular, primitively embeds in $N$, hence in $\mathbb{L}_{K3}$. One can find some explicit conditions that

[5] The Lie algebra of $\mathrm{SO}(N_\mathbb{R})$ can be identified with $\Lambda^2 N_\mathbb{R}$ so that $\Lambda^2 J$ can be identified with a subalgebra of the Lie algebra. Also $(I^\perp/I)_\mathbb{R}$ can be identified with a subalgebra of the Lie algebra by choosing a generator $f$ of $I_\mathbb{R}$ and sending $x\in(I^\perp/I)_\mathbb{R}$ to $f\wedge x\in\Lambda^2 N_\mathbb{R}$.

[6] Recall that for any real affine space $V$ over a linear space $L$, a tube domain is a subset of $V_\mathbb{C}$ of the form $V+iC$, where $C$ is a cone in $L$ not containing lines.

guarantee that an isotropic vector $f$ is $m$-admissible for some $m$ (see [2], Prop. 5.5) and also that the primitive embedding of $I^\perp/I$ in $N$ obtained in this way is unique up to an isometry of $N$ extending to an isometry of $\mathbb{L}_{K3}$.

Let us assume that $I$ is generated by a $m$-admissible isotropic vector $f$ and fix a primitive embedding of $I^\perp/I$ in $N$. We denote the image by $I^\perp/I$ by $\check{M}$ (the right notation should be $\check{M}_I$ since its definition depends on $I$). It is a primitive sublattice of $N$ and its signature is equal to $(1, 19-\rho)$.

Thus we get

$$N \cong U(m) \oplus \check{M},$$

and

$$\mathcal{D}_N^o \cong \check{M}_{\mathbb{R}} + iC_{\check{M}}.$$

Since isometry $\mathrm{id}_{U(m)} \oplus -\mathrm{id}_{\check{M}}$ belongs to $\mathrm{O}(N)^*$ and it switches the two connected components of $\mathcal{D}_N$, we obtain that $\mathcal{M}_{K3,M}$ is an irreducible quasi-projective variety, the quotient of a connected component of $\mathcal{D}_N$ by a subgroup $\mathrm{O}(N)_0^*$ of index 2 of the group $\mathrm{O}(N)^*$. We also choose a fundamental domain $\Pi_{\check{M}}$ of the 2-reflection group $W_{\check{M}}$ in $C_{\check{M}}$. Now we may define the *mirror moduli space* $\mathcal{M}_{K3,\check{M}}$. Take note that the definition of the mirror moduli space depends on the choice of 0-dimensional rational boundary component $F_I$ defined by an isotropic line $I$ in $N$. It may not exist so the moduli space is compact in this case. Also, note that, if $m = 1$, then

$$\check{N} := \check{M}^\perp_{\mathbb{L}_{K3}} = M \oplus U,$$

so we may define the mirror moduli space of $\mathcal{M}_{K3,\check{M}}$ with respect an isotropic line contained in $U$, and it coincides with $\mathcal{M}_{K3,M}$. This explains why the construction is called the *mirror symmetry*.

Let $\mathcal{D}_N^*$ be the union of $\mathcal{D}_N$ and rational boundary components defined by isotropic lattices in $N$. As in the case of elliptic curves, one defines a topology on $\mathcal{D}_N^*$ and defines a structure of an analytic space on the orbit space $\mathcal{D}_N^*/\mathrm{O}(N)^*$ that makes it into a projective algebraic variety $\overline{\mathcal{D}_N/\mathrm{O}(N)^*}$. The group $\Gamma = \mathrm{O}(N)^*$ acts on $\mathcal{D}_N^*$. Let $F_I$ be a 0-dimensional rational boundary component and $\Gamma_I$ be its stabilizer subgroup. We assume that $I = \mathbb{Z}f$, where $f$ is $m$-admissible and denote by $g$ an isotropic vector with $f \cdot g = m$. The group $\Gamma_I$ is equal to the stabilizer of $I$ in the index 2 subgroup of $\Gamma_0$ of $\Gamma$ that preserves the connected component of $\mathcal{D}_N$ containing $F_I$. Let $\overline{\Gamma_I}$ be its image in $\mathrm{O}(I^\perp/I) = \mathrm{O}(\check{M})$, it is easy to see that $\overline{\Gamma_I}$ is equal to a subgroup of finite index of $\mathrm{O}(\check{M})$. It fits in the split group extension:

$$0 \to \check{M} \xrightarrow{\iota} \Gamma_I \xrightarrow{r} \overline{\Gamma}_I \to 1,$$

where the homomorphism $\iota : \check{M} \to \Gamma_I$ is given by

$$\iota(v)(w) = w + \frac{1}{m}(w \cdot f)v - \big(w \cdot v + \frac{1}{2m}(w \cdot f)v^2\big)f. \tag{2}$$

It is immediately checked that the definition of $\iota(v)$ depends only on the coset $v + I \in I^\perp/I$. Also, if $w \in I^\perp$, then $\iota_v(w) \equiv w \mod I$, so that the image of $\iota$ belongs to the kernel of $r$. We denote the image of $\check{M}$ under this map by $\Gamma^I$. One can express the existence of the group extension by saying that $\Gamma_I \cong \Gamma^I \rtimes \overline{\Gamma}_I$. The group $\Gamma^I$ is the analog of the group $\Gamma_\infty$ in the elliptic case. Recall that the quotient $U_c/\Gamma_\infty$ of the subset of $\tau \in \mathbb{H}$ with $\mathrm{Im}(\tau) > c > 1$ is isomorphic to a neighborhood in $X(\Gamma)$ of the cusp corresponding to the rational boundary point $\infty$. In our case, the same is true. Let $U$ be the tube domain $\check{M}_\mathbb{R} + iC_{\check{M}}$. The group $\Gamma^I \cong \check{M}$ acts on $U$ and the quotient is an open subset $V$ of the algebraic torus $\mathbb{T} = \check{M}_\mathbb{C}/\check{M} = \check{M}_\mathbb{R}/\check{M} + i\check{M}_\mathbb{R}$. Let $\mathrm{ord} : \mathbb{T} \to \check{M}_\mathbb{R}$ be the projection $x + iy \mapsto y$. The open set $V$ is equal to $\mathrm{ord}^{-1}(C_{\check{M}})$. We can choose a basis $(\alpha_1, \ldots, \alpha_{20-\rho})$ in the dual lattice $\check{M}^\vee \subset \check{M}_\mathbb{R}$ such that $C_{\check{M}}$ is contained in the set of vectors with positive coordinates, thus the projection $U \to V$ can be given by the map

$$\exp : U \to \mathbb{T} \cong (\mathbb{C}^*)^{20-\rho}, \quad z = x + iy \mapsto (e^{2\pi i(z,\alpha_1)}, \ldots, e^{2\pi i(z,\alpha_{20-\rho})}).$$

It is clear that $V$ is contained in the polydisk $(\Delta^*)^{20-\rho}$, where $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

The lattice $\check{M}$ is identified with the lattice of one-parameter subgroups of the torus $T$ (the $N$-lattice from the theory of toric varieties). Let $(\sigma_\alpha)_\alpha$ be a $\overline{\Gamma}_I$-invariant polyheddral decomposition of $C$ forming an infinite fan $\Sigma$ in $\check{M}_\mathbb{R}$. Let $\mathbb{T} \subset X_\Sigma$ be the corresponding *toric embedding*. An example of such a fan is the set of closures of fundamental domains of the 2-reflection subgroup $W_{\check{M}}$. For every $c \in C = C_{\check{M}}$, set

$$C_c = C + c \subset C,\ U_c = \mathrm{ord}^{-1}(C_c),\ V_c = U_c/\check{M}.$$

Let

$$V' = V \cup (X_\Sigma \setminus \mathbb{T}), \quad V_c' = V_c \cup (X_\Sigma \setminus \mathbb{T})$$

be the interior of the closure of $V, V_c$ in $X_\Sigma$. One can show (see [1], III, Theorem 1.4) that $\overline{\Gamma}_I$ acts properly discontinuously on $V'$ and $\overline{\Gamma}_I \cdot V_c'$ is open and relatively compact in $V/\overline{\Gamma}_I$. One can also prove that for $c$ with large enough norm $c^2$, $\overline{\Gamma}_I \cdot V_c/\overline{\Gamma}_I$ is mapped isomorphically into $U/\Gamma$. Now we compactify $U/\Gamma$ by gluing $U/\Gamma$ and $\overline{\Gamma}_I \cdot V_c'$ along the set $\overline{\Gamma}_I \cdot V_c$. we do it for each 0-dimensional component. Assuming that there are 1-dimensional rational boundary components, we get in this way a *toroidal compactification* of $\mathcal{D}_N/\mathrm{O}(N)^*$.

In order to take into account one-dimensional rational boundary components, we proceed as follows. Let $F_J$ be a one-dimensional rational boundary component corresponding to an isotropic plane $J$ in $N$. We consider the projections

$$\pi_J : \mathbb{P}(N_\mathbb{C}) \dashrightarrow \mathbb{P}((N/J)_\mathbb{C}) \cong \mathbb{P}^{19-\rho}, \quad \pi_{J^\perp} : \mathbb{P}(N_\mathbb{C}) \dashrightarrow \mathbb{P}((N/J^\perp)_\mathbb{C}) \cong \mathbb{P}^1.$$

Restricting the projections to a conenected component $\mathcal{D}_N^o$ of $\mathcal{D}_N$, we obtain holomorphic maps

$$\mathcal{D}_N^o \to \pi_J(\mathcal{D}_N^o) \to \pi_{J^\perp}(\mathcal{D}_N^o).$$

By taking an isotropic line $I \subset J$, we see that the fibers of the first projection are isomorphic to the upper half-planes and the target space $\pi_{J^\perp}(\mathcal{D}_N^o)$ is isomorphic to the half-plane $\mathbb{H}$. To see this, let us choose an isotropic plane in $N_{\mathbb{R}}$ with a basis $(g,g')$ such that $f\cdot g=1, f'\cdot g'=1$ and the hyperbolic planes $H=\mathbb{R}f+\mathbb{R}g, H'=\mathbb{R}f'+\mathbb{R}g'$ are orthogonal. We can identify $(J^\perp/J)_{\mathbb{R}}$ with $(H\oplus H')^\perp$. Then the points of $\pi_I(\mathcal{D}_N)$ can be written in the form $z=(x_0f'+x_0'g'+x)+i(y_0f'+y_0'g'+y), 2y_0y_0'+y^2>0, y_0>0$, where $x,y\in(H\oplus H')^\perp$. The projection $\pi_I(\mathcal{D}_N^o)\to\pi_J(\mathcal{D}_N^o)$ is given by $z\mapsto(x_0'g'+x)+i(y_0'g'+y)$ and its fibers are isomorphic to the upper half-plane of complex numbers $x_0+iy_0, y_0>0$. The target space consists of vectors $x_0'g'+iy_0'g'$ such that $2y_0y_0'>-y^2>0$, hence $y_0'>0$. It is isomorphic to the upper-half plane. The fibers of the projection $\pi_{J^\perp}$ are affine spaces isomorphic to the linear space of vectors $x_0f'+x+i(y_0f'+y)$. Its dimension is equal to $19-\rho$.

Let $\Gamma_J$ be the stabilizer subgroup of $J$ in $\mathrm{O}(N)^*$, it is equal to the stabilizer of the boundary component $F_J$. Let $\Gamma^J$ be the kernel of the natural homomorphism $\Gamma_J\to\mathrm{GL}(J)$. The image is a subgroup of finite index of $\mathrm{SL}(J)\cong\mathrm{SL}_2(\mathbb{Z})$ (we use that $\Gamma_J$ preserves the orientation of $J$. The group $\Gamma^J$ contains a subgroup $\Gamma_0^J$ of finite index that acts identically on $J^\perp/J$ (recall that $J^\perp/J$ is a negative definite lattice and $\Gamma^J$ is mapped to its orthogonal group).

For any element $g$ in $\Gamma_0^J$, the restriction of $g-1$ to $J^\perp$ induces a linear map $\phi: J^\perp\to J$ that is identically zero on $J\subset J^\perp$. This defines a homomorphism

$$\Gamma_0^J\to\mathrm{Hom}(J^\perp/J,J)\cong J\otimes J^\perp/J,$$

where we identify $J^\perp/J$ with its dual space using the non-degenerate symmetric bilinear form on $J^\perp/J$. One can show that this homomorphism is surjective, and the group $\Gamma_1^J$ fits in the extension

$$1\to\Lambda^2J\to\Gamma_1^J\to J\otimes J^\perp/J\to 1,$$

where the first non-trivial homomorphism is given by sending $u\wedge v$ to the transformation

$$t_{u,v}: w\mapsto w+(w,u)v-(w,v)u.$$

The subgroup $\Lambda^2J\cong\mathbb{Z}$ is the center, and the quotient $J\otimes J^\perp/J$ is a free abelian group of rank $2(18-\rho)$. This makes $\Gamma^J$ isomorphic to a group of integer points of a real *Heisenberg group*.

Consider the quotient $\mathcal{D}_N^o/\Gamma_J$. First we divide by $\Gamma_1^J$ and then divide $\mathcal{D}_N^o$ by $\Gamma_J/\Gamma_1^J$. The center $Z_J\cong\Lambda^2J$ of $\Gamma_1^J$ preserves the half-plane fibration $\mathcal{D}_N^o\to\pi_J(\mathcal{D}_N^o)$ and the quotient becomes isomorphic to the punctured disk fibration $\mathcal{D}_N^o/Z_J\to\pi_J(\mathcal{D}_N^o)$. The fibers of $\pi_J(\mathcal{D}_N^o)/\Gamma_1^J\to\pi_{J^\perp}(\mathcal{D}_N^o)$ are isomorphic to complex tori of dimension $18-\rho$. In fact, the alternating form on $J\otimes J^\perp/J$

$$(J\otimes J^\perp/J)\times(J\otimes J^\perp/J)\to J\times J\to\Lambda^2J,$$

where the pairing is defined by the symmetric bilinear form on $J^\perp/J$, defines a polarization on the fibers, so that we have an abelian fibration over the upper-half plane $\pi_{J^\perp}(\mathcal{D}_N^o)$. Let $L$ be the line bundle on $\pi_J(\mathcal{D}_N^o)$ defined by this polarization. One checks that the punctured disk fibration $\mathcal{D}_N^o/Z_J \to \pi_J(\mathcal{D}_N^o)$ sits in the total space $\mathbb{L}^*$ of $L$ minus the zero section. We close the punctured disk fibration in the total space $\mathbb{L}$ of $L$ adding a divisor $D$ isomorphic to the abelian fibration over the upper-half plane. A small open neighborhood of $D$ in $\mathbb{L}$ minus $D$ is isomorphic to an open neighborhood of the image of $F_J$ in the compactification $\mathcal{D}_N^*/\emptyset(N)^*$. It is glued to toroidal compactification at the images of $F_I, I \subset J$. It remains to divide by $\Gamma_J/\Gamma^J$. The result is a projective algebraic variety $\overline{\mathcal{D}_J/\mathrm{O}(N)^*}$ completing $\mathcal{D}_J/\mathrm{O}(N)^*$. Any $\mathrm{O}(N)^*$-orbit of a one-dimension rational boundary component $F_J$ defines a codimension 1 subvariety $Y_J$ of the boundary. It is isomorphic to a finite quotient of an abelian fibration over the modular curve $X(\Gamma(J))$, where $\Gamma(J)$ is the image of $\Gamma_J$ in $\mathrm{SL}(J)$. [7] Over each cusp of $X(\Gamma(J))$ we have a singular fiber which is defined by the toroidal compactification corresponding to a cusp on $X(\Gamma(J))$ associated to the $\mathcal{O}_N^*$-orbit of an isotropic line $I$ contained in $J$. If $I$ is not contained in any isotropic plane, then it defines a codimension 1 subvariety of the boundary isomorphic to the fiber of the toroidal compactification over the corresponding point in $\mathcal{D}^*/\mathrm{O}(N)^*$.

This describes a toroidal compactification $\mathcal{M}_{K3,M}^{\mathrm{tor}}$ of $\mathcal{M}_{K3,M}$. Note that there is a morphism $\mathcal{M}_{K3,M}^{\mathrm{tor}} \to \mathcal{M}_{K3,M}^{\mathrm{BB}}$, where $\mathcal{M}_{K3,M}^{\mathrm{BB}}$ is the *Bailey-Borel* compactification. Its boundary consists of the union of open modular curves $X'(\Gamma) = X(\Gamma(J)) \setminus \{cusps\}$ and images of 0-dimensional boundary components (the closures of $X'(\Gamma)$ in $\mathcal{M}_{K3,M}^{\mathrm{BB}}$ could be singular at cusps).

## 4. The moduli of polarized K3 surfaces and its mirror

We consider the simplest special case when the lattice $M$ is of rank 1. Let $e$ be its generator and $e^2 = 2n$. A K3 surface $(X, j)$ with an ample $M$-polarization is called a K3 surface of *genus* $g = n+1$. The reason for this confusing terminology is that a nonsingular member of the linear system $|j(e)|$ is a curve of genus $n+1$. If $n > 1$, an ample $M$-polarization on $X$ defines a very ample complete linear system $|kj(e)|, k \geq 3$, that embeds $X$ in $\mathbb{P}^{k^2n+1}$ as a surface of degree $2kn$. If the polarization is not ample, then $|kj(e)|, k \geq 3$, defines a birational morphism onto a surface $\bar{X}$ of degree $2kn$ in $\mathbb{P}^{k^2n+1}$ that contains rational double points, the images of smooth rational curves $C$ on $X$ with $C \cdot j(e) = 0$.[8] For any smooth family $f: \mathcal{X} \to \mathsf{S}$ of $M$-polarized K3 surfaces, there is a morphism $\mathsf{S} \to \mathrm{Hilb}^P(\mathbb{P}^{k^2n+1})$ to the Hilbert scheme of closed subschemes of $\mathbb{P}^{k^2n+1}$ with Hilbert polynomial $P(t) = knt^2 + 2$. The image is contained in an open subset $U$ of an irreducible component $\mathrm{Hilb}_0^P(\mathbb{P}^{k^2n+1})$ of dimension $19 + \dim \mathrm{PGL}(k^2n+2)$. The open

[7] In fact, the abelian fibration is isomorphic to the fiber of $18-\rho$ copies of the modular elliptic surface (assuming that $-1 \notin \Gamma(J)$.

[8] If $X$ has no smooth curves of genus 1 intersecting $h$ with multiplicity $\leq 2$, then we may take $k = 1$.

subset is contained in the set of stable points with respect to the action of $\mathrm{SL}(k^2n+2)$, and the quotient $\mathcal{F}_g$ is a quasi-projective variety playing the role of a coarse moduli space of $M$-polarized K3-surfaces (see [14]). The theory of periods defines an isomorphism $\mathcal{F}_{n+1} := \mathcal{M}_{K3,M}$.

One can show that all primitive embedding $\langle 2n\rangle \hookrightarrow \mathbb{L}_{K3}$ are equivalemt with respect to $\mathrm{O}(L)$. Thus we may assume that a generator $e$ of $\langle 2n\rangle$ embeds into the hyperbolic plane orthogonal summand $U$ of $\mathbb{L}_{K3}$ with standard basis $a, b$ as $e = a + nb$. Then, it follows that

$$N = (M^\perp)_{\mathbb{L}_{K3}} \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2n\rangle,$$

where for any integer $m$, we denote by $\langle m\rangle$ the lattice $\mathbb{Z}v$ with $v^2 = m$. We use $f, g$ and $f', g'$ (resp. $t$) to denote the standard bases of the two copies of the hyperbolic plane orthogonal summands of $N$ (resp. $\langle -2n\rangle$).

Let us look at the compactifications of $\mathcal{F}_g^{\mathrm{tor}}$ and $\mathcal{F}_g^{\mathrm{BB}}$ of $\mathcal{F}_g$. The set of 0-dimensional boundary components in $\mathcal{F}_g^{\mathrm{BB}}$ is bijective to the set $\mathcal{I}_1(N)$ of $\mathrm{O}(N)^*$-orbits of primitive rank 1 isotropic sublattices $I \subset N$. It is known that the number of orbits is equal to $[\frac{m+2}{2}]$, where $n = km^2$ and $k$ is square-free (see [11], Theorem 4.0.1). Let $f$ be a primitive isotropic vector, the map $N \to \mathbb{Z}, x \mapsto x \cdot f$ has the image a cyclic group generated by an integer which we denote by $\mathrm{div}(f)$ and, if $I = \mathbb{Z}f$, we set $\mathrm{div}(I) = \mathrm{div}(f)$. Obviously, $\mathrm{div}(I) = \mathrm{div}(I')$ if $I$ and $I'$ belong to the same orbit. The discriminant group $A_N = N^\vee/N$ of the lattice $N$ is isomorphic to $\langle \frac{1}{2n}t\rangle$, and the map $f \mapsto \frac{1}{\mathrm{div}(f)}f + N$ is a bijection from the $\mathcal{I}_1(N)$ to the set of isotropic vectors in $A_N$ modulo multiplication by $\pm 1$. An element $x = \frac{a}{2n}t + N \in A_L$ is isotropic if and only if $q(x) = -a^2 2n/4n^2 = -a^2/2n \in 2\mathbb{Z}$. Each isotropic element in $A_N$ generates an isotropic subgroup of $A_N$. Let $y = \frac{1}{d}t + N, d|2n$ be its generator. Then $d|2n, d^2|4n$ implies that $d|m$. In particular, we see that, for any primitive isotropic vector in $N$, we have $\mathrm{div}(f)|m$. This shows that the the set of isotropic elements in $A_N$ is equal to the number of divisors of $m$.

Next we look at the set of 1-dimensional boundary components of $\mathcal{F}_g^{\mathrm{BB}}$. It is bijective to the set $\mathcal{I}_2(N)$ of primitive isotropic rank 2 sublattices of $N$. For each sublattice $J$, we look at the negative definite lattice $J^\perp/J$. We have $(J^perp/J)_\mathbb{Q} \cong (E_8^{\oplus 2} \oplus \langle -2n\rangle)_\mathbb{Q}$. Recall that two definite quadratic lattices belong to the same *genus* if they are isomorphic over all rings of $p$-adic numbers $\mathbb{Z}_p$ and over $\mathbb{R}$. Each genera contains a finite number of isomorphism classes of lattices. Let $h(n)$ be the number of isomorphism classes of the genera $\mathcal{G}_k$ of the lattice $E_8^{\oplus 2} \oplus \langle -2n\rangle$. For example, when $h(1) = 4$, the isomorphism classes in $\mathcal{G}_k$ are represented by four lattices unqely determined by their sublattice, generated by vectors of norm $-2$ (*root sublattice*):

$$E_8^{\oplus 2} \oplus \langle -2\rangle,\ D_8^{\oplus 2} \oplus \langle -2\rangle,\ D_{16} \oplus \langle -2\rangle,\ A_{17}. \tag{3}$$

For example, the first type is realized when we take $J$ to be generated by $f$ and $f'+g'+t$. The second type is realized when we take $J$ to be generated by $f$ and $f'+g'+v$, where $v$ belongs to a summand $_8$ and $v^2=-2$ (resp. bThe genera $\mathcal{G}_2$ is represented by nine isomorphism classes of lattices uniquely determined by their sublattic generated by vectors with norm $-2$ and $-4$:[9]

$$E_8^{\oplus 2}\oplus\langle -4\rangle,\ D_8^{\oplus 2}\oplus\langle -4\rangle,\ D_{16}\oplus\langle -4\rangle, E_8\oplus D_9 \tag{4}$$
$$E_7^{\oplus 2}\oplus A_3,\ D_{17},\ D_{12}\oplus D_5,\ A_1^{\oplus 2}\oplus A_{16},\ E_6\oplus A_{11}. \tag{5}$$

A lattice $J\in\mathcal{I}_2(N)$ defines an invariant with respect to the action of $\mathrm{O}(L)^*$ that is analogous to $\mathrm{div}(f)$ defined in above. We consider the orthogonal complement of $(J^\perp)^\perp_{N^\vee}$ in $N^\vee$, i.e. the subgroup of $N^\vee$ of linear functions that vanish on $J^\perp$. Its intersection with $N$ consists of $J$, hence $(J^\perp)^\perp_{N^\vee}/J$ embeds in $A_L=L^\vee/L\cong\mathbb{Z}/2n\mathbb{Z}$ as a cyclic group of some order $e$. Since its generator is an isotropic vector with respect to the discriminant form on $A_L$, we obtain that $e^2|n$. We recycle the notation and denote $e$ by $\mathrm{div}(J)$. If $e=1$, then the set $\mathcal{I}_{2,1}(E)/\mathrm{O}(N)^*$ is bijective to the set $\mathcal{G}(k)$.

Let $\mathcal{I}_{2,e}(N)$ denote the subset of $\mathcal{I}_2(N)$ of $J'$ with fixed $e$. It is proven in [11] that the open modular curves $X'(\Gamma)$ in $\mathcal{F}_g^{\mathrm{BB}}$ corresponding to a boundary component $F_J$ with $\mathrm{div}(J)=e$ is isomorphic to the curve $X_1(e)'=X_1(e)\setminus\{cusps\}$. The cusps are nonsingular points on the closure of $X_1(e)'$ if and only if $e=1$ or $e=3$ (see [11], 5.0.3).

For, example, assume $n=1$, then we have four modular curves $X_1(1)\cong X(1)$ intersecting at one cusp (recall that the modular curve $X(\Gamma(1))$ has only one cusp). If $n=2$, then we have 1 cusp and 9 modular curves intersecting at the cusp.

Let us now look at the mirror moduli spaces. Let $n=km^2$ be as above. We know that there are $[\frac{k+2}{2}]$ mirror families, each depending on a choice of $I=\mathbb{Z}f\in\mathcal{I}_1(N)/\mathrm{O}(N)^*$. Let $d=\mathrm{div}(I)$. We know that $d|m$. Let $v=\alpha e+d(f+\frac{\alpha^2 n}{d^2}g)\in N$ and $v^2=0, v\cdot g=d=\mathrm{div}(v)$. Also $v$ is a primitive isotropic vector if we choose $\alpha$ coprime to $d$. Let $U(d)=\mathbb{Z}f+\mathbb{Z}g$. Then $I^\perp/I\cong U(d)^\perp$. The lattice $U(d)$ is contained in $U\oplus U\oplus\mathbb{Z}e$, and it is immediate that $U(d)^\perp\cong E_8^{\oplus 2}\oplus\langle -2n/d\rangle$. Thus the lattice $\check{M}$ (which depends on a choice of $I$) is isomorphic to the lattice

$$M_{n/d}:=E_8^{\oplus 2}\oplus\langle -2n/d\rangle.$$

So, $\mathcal{F}_{n+1}=\mathcal{M}_{K3,\langle 2n\rangle}$ has $[m+2/2]$ mirror moduli spaces each isomorphic to some $\mathcal{M}_{K3,M_{n/d}}$, where $d^2|n$.

Let us look at the moduli space $\mathcal{M}_{K3,M_n}=\mathcal{D}_N/\Gamma_N^*$ more closely. First, $N=U\oplus\langle 2n\rangle$ is of rank 3, hence $\dim\mathcal{D}_N=1$. Its connected component $\mathcal{D}_N^o$ in its tube domain realization must coincide with the upper half-plane. Another way to see it is to use that the quadric $Q_N$ is a conic in $\mathbb{P}(N_{\mathbb{C}})\cong\mathbb{P}^2$

[9] On p. 2623 in [2] we incorrectly identified the lattices in (3) and (**??**) with the sublattices $J^\perp/J$.

which is isomorphic to $\mathbb{P}^1(\mathbb{C})$ under the Veronese embedding. Take the basis $(f, g, e)$ in $N$ such that the quadratic form can be written in this basis as $q = 2xy - 2nz^2$. One can identify it with the discriminant of a binary form $xU^2 + 2\sqrt{n}zUV + yV^2$ multiplied by $-2$. The group $\mathrm{SL}_2(\mathbb{R})$ acts naturally on the space of such quadratic forms via a linear coordinate change. Obviously, it preserves the discriminant, hence defines a homomorphism $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}(N_\mathbb{R}) \cong \mathrm{SO}(2,1)$. Its kernel is $\{\pm 1\}$ and the image is $\mathrm{SO}_0(2,1)$, the subgroup that preserves a connected component of $\mathcal{D}_N$. Thus we obtain an isomorphism $\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SO}_0(2,1)$. We have a canonical injective homomorphism $\mathrm{O}(N)_0^* \to \mathrm{SO}(2,1)$, hence we obtain an injective homomorphism $\mathrm{O}_0(N)^* \to \mathrm{PSL}_2(\mathbb{R})$. Explicit computations in [2], Theorem 7.1, give an isomorphism

$$\mathrm{O}_0(N)^* \cong \Gamma_0(n)^+,$$

where $\Gamma_0(N)^+$ is generated by the group $\Gamma_0(n)$ and the *Fricke involution* $F = \begin{pmatrix} 0 & -1 \\ n & 0 \end{pmatrix}$. Thus we obtain

$$\mathcal{M}_{K3,M_n} \cong \mathbb{H}/\Gamma_0(n)^+, \ \overline{\mathcal{M}}_{K3,\check{M}_n} \cong X_0(n)^+ := X_0(n)/\langle F \rangle.$$

If $n = 1$, the group $\Gamma_0(n) = \Gamma(1)$ acts with the kernel $\pm 1$. For $n \geq 2$, the curve $\mathbb{H}/\Gamma_0(n)^+$ is isomorphic to the fine moduli space of pairs $(E, H)$, where $E$ is an elliptic curve and $H$ is a cyclic subgroup of $E$ order $n$. The Fricke involution sends $(E, H)$ to $(E', H')$, where $E' = E/H$ and $H' = H^\perp$ is taken with respect to the Weil pairing on $E[n]$. The isomorphism between the moduli spaces of K3 surfaces and $X_0(n)/\langle F \rangle$ is defined by considering the Kummer surface associated to the abelian surface $E \times E'$ (see Appendix).

*Remark* 2. The genus of $X_0(n)$ is equal to zero if and only if $n = 2-10, 12, 13, 16, 18, 25$ [3], p. 304, [10]. The genus of $X_0(n)^+$ is equal to zero for a larger set of $n$. The prime $n$ entering in this set are the prime divisors of the order of the Monster group.

One can compute the number of points on $X_0(n)^+$ corresponding to isomorphism classes of non-amply $M_n$-polarized K3 surfaces. For $n \geq 5$ they are the branch points of the double cover $X_0(n)^+ \to X_0(n)$, or, equivalently, the images of the fixed points of the Fricke involution on the curve $X_0(n)$ under the cover $X_0(n)^+ \to X_0(n)$. The number was computed by R. Fricke [3], and it is equal to $h_0(-n) + h_0(-4n)$ if $n \equiv 2, 3 \mod 4$, and $h_0(-4n)$ if $n \equiv 1 \mod 4$. Here $h_0(-d)$ is the class number of primitive quadratic integral positive definite forms with discriminant equal to $-d$. If $n \leq 4$, there is only one such point. The curve $X_0(n)$ is isomorphic to $\mathbb{P}^1$, and there are two ramification points. If $n = 1, 2, 3$, then $F$ fixes two points, one of them is the unique $\Gamma_0(2)$-orbit of the point $i = \sqrt{-1}$ with stabilizer of order 2. The other fixed point represents the unique non-amply $M_n$-polarized K3 surface. If $n = 4$, then one of the fixed points of $F$ is a cusp of width 2.

**Example 3.** Let $n = 2$. Consider the Hesse pencil of quartic surfaces:

$$X(\lambda) := x_0^4 + x_1^4 + x_2^4 + x_3^4 - \lambda x_0x_1x_2x_3 = 0. \tag{6}$$

The surface is nonsingular if $\lambda^4 \neq 1/4$, otherwise it has 16 ordinary double points. The group $G = \langle g_1, g_2\rangle = (\mathbb{Z}^4)^{\oplus 2}$ acts in $\mathbb{P}^3$ by $g_1, g_2 : [x_0, x_1, x_2, x_3] \mapsto [ix_0, -ix_1, x_2, -ix_3], [ix_0, x_1, -ix_2, x_3]$. The subring of invariants of this group in $\mathbb{C}[x_0, x_1, x_2, x_3]$ is generated by $u_i = x_i^4$ and $u_4 = x_0x_1x_2x_3$ satisfying $u_4^4 = u_0u_1u_2u_3$. Thus the quotient $X(\lambda)$ is isomorphic to the quartic surface $Y(\lambda)$ in $\mathbb{P}^4$ given by the equations

$$u_4^4 = u_0u_1u_2u_3, \quad u_0 + u_1 + u_2 + u_3 + \lambda u_4 = 0. \tag{7}$$

Eliminating $u_0$, we obtain a model of this surface in $\mathbb{P}^3$:

$$u_4^4 - u_1u_2u_3(u_1 + u_2 + u_3 + \lambda u_4) = 0.$$

If $\lambda^4 \neq 1/4$, it has 6 singular points of type $A_3$ with coordinates $u_4 = u_i = u_j = 0$ or $u_4 = u_i = u_1 + u_2 + u_3 = 0$. If $\lambda^4 = 1/4$, it has an additional ordinary double point. The surface $Y(\lambda)$ contains 4 lines with equations $u_4 = u_i = 0$ and $u_4 = u_0 + u_1 + u_2 + u_3 = 0$, its intersection points are the singular points. Consider the pencil of planes $H_t : t_0u_4 - t_1(u_1 + u_2 + u_3) = 0$ passing through one of the lines, say $u_1 + u_2 + u_3 = u_4 = 0$. The residual curves of the plane sections are plane cubics with equations $t_1^4u_4^3 + t_0^3(t_1\lambda + t_0)u_1u_2u_3 = 0$. This defines an elliptic fibration on $X(\lambda)$ with at least 3 sections coming from the base points of this pencil. It has two reducible fibers of Kodaira's types $IV^* = \tilde{E}_6$ and $I_{12} = \tilde{A}_{11}$ over 0 and $\infty$. Let $d$ be the discriminant of $S_{X(\lambda)}$ and MW be the Mordell-Weil group of sections of the ellptic fibration. By Shioda-Tate's Formula, we have $d|\mathrm{MW}|^2 = 12 \cdot 3$. This gives $d = 4$. In fact, it also gives that $A_{S(X(\lambda))} = \mathbb{Z}/4\mathbb{Z}$. It follows from the theory of quadratic forms that there is only one isomorphism class of lattices of signature $(1, 18)$ and the discriminant group $\mathbb{Z}/4\mathbb{Z}$. It is isomorphic to our lattice $\check{N} = U \oplus E_8 \oplus E_8 \oplus \langle -4\rangle$. Thus we constructed a family of $\check{M}_2$-polarized K3 surfaces $f : \mathcal{X} \to \mathsf{S} = \mathbb{P}^1$. Note that the period map per : $S \to \mathcal{M}_{K3,\check{M}_2} = X_0(2)^+$ is not bijective. The subgroup $\mu_4$ of $\mathbb{C}^*$ acts on the total family and its base by $\lambda \to c\lambda$, defining a family $f' : \mathcal{X}' = \mathcal{X}/\mu_4 \to \mathsf{S}/\mu_4 = \mathbb{P}^1$. One can show that the new period map $\mathrm{per}' : \mathbb{P}^1 \to X_0(2)^+$ is an isomorphism (one checks that over the unique cusp of $X_0(2)^+$ the map is an isomorphism.

Consider the neighborhood of the cusp. The family defines a map over a disk $Y \to \Delta$. Over a cyclic cover of degree 4, this family is birationally isomorphic to the Hesse pencil $X(\lambda)$ in the neighborhood of the point $\lambda = \infty$. The total family has singular points at the intersection of $V(x_0^4 + \ldots + x_3^4)$ with the coordinate lines $x_i = x_j = 0$. Its singular fiber is the union of 4 planes. One can birationally transform the family to assume that $Y$ is smooth and the relative canonical class is trivial, and the singular fiber in the *minus-one* form, i.e. the self-intersection of each double curve is equal to $-1$ on the corresponding irreducible component. The dual polyhedron is a tetrahedron. The surface is the union of 4 cubic surfaces glued together along tritangent plane sections.

In our example, there are 5 singular members of the pencil (7) corresponding to the values of $\lambda^4$ equal to $\infty$ and $1/4$. On the quotient by $\mu_4$ there are only two points. One of them correspond to type III degeneration which we considered before, another one corresponds to the unique non-amply $M_2$-polarized K3 surface.

*Remark* 4. We have found two different elliptic pencils on a $M_2$-polarized K3 surface $X$ with reducible fibers of types $II^*, II^*, I_1$ and $I_{11}, IV^*$. Suppose the Picard lattice $S_X$ contains a primitive sublattice isomorphic to $U \oplus R$, where $R$ is generated by vectors with norm equal to $(-2)$ (hence the direct sum of root lattices of finite type). Then there exists an elliptic fibration on $X$ with a section and reducible fibers described by affine Dynkin diagrams corresponding to direct irreducible summands of $R$ (see [6], Lemma 2.1). Let us apply this to our case when $S_X = M_n$. For any primitive sublattice $J \in \mathcal{I}_{2,1}$ of $N = \langle 2n\rangle^\perp = M_n \oplus \langle -2n\rangle$, $J^\perp/J$ is isomorphic to a sublattice of $M_n$ and contains a hyperbolic plane $U$ in its orthogonal complement. Let $R_J$ be the sublattice of $J^\perp/J$ generated by vectors of norm $-2$. Then $U \oplus (J^\perp/J)'$ defines an elliptic fibration on $X$ with reducible fibers of types defined by $R_J$.

For example, when $n = 1$ (resp. $n = 2$) we get from (3) (resp. (4)) that $X$ has elliptic fibrations with reducible fibers of type

$$II^*, II^*, I_2;\ I_4^*, I_4^*, I_2;\ I_012^*, I_2;\ I_{18},$$

(resp.

$$II^*, II^*;\ I_4^*, I_4^*;\ I_{12}^*;\ II^*, I_5^*;\ III^*, III^*, I_4;\ I_{13}^*;\ I_8^*, I_1^*, I_1^*;\ IV^*, I_{12}; I_{16}, I_2, I_2)$$

## 5. Appendix:Shioda-Inose structure

Recall the following facts about K3 surfaces with large Picard number (see [9]).

**Theorem 5.** *Let $X$ be a complex algebraic K3 surface. The following properties are equivalent.*

(i) *There exists an abelian surface $A$ and an isometry $T_X \cong T_A$ preserving the Hodge decomposition;*
(ii) *There is a primitive embedding $T_X \hookrightarrow U \oplus U \oplus U$;*
(iii) *There is a primitive embedding $E_8 \oplus E_8 \hookrightarrow S_X$;*
(iv) *There exists an involution $\sigma : X \to X$ such that $X/(\sigma)$ is birationally isomorphic to the Kummer surface $Kum(A) = A/(a \mapsto -a)$;*

Let us sketch the proofs of some of these implications. (i) $\Rightarrow$ (ii) For any abelian surface $A$, $H^2(A, \mathbb{Z})$ is a unimodular even indefinite lattice of rank 6. By Milnor's Theorem it must be isomorphic to $U^{\oplus 3} := U \oplus U \oplus U$. Thus (i) implies that there exists a primitive embedding $T_X \hookrightarrow U^{\oplus 3}$.

(ii) $\Rightarrow$ (iii) We have $T_X \hookrightarrow \mathbb{L}_{K3} = U^{\oplus 3} \oplus E_8 \oplus E_8$. One can show, using Nikulin's results from [7], that all embedding of a lattice of rank $\le 6$ and signature $(2,1)$ in $\mathbb{L}_{K3}$ are equivalent under an isometry of $\mathbb{L}_{K3}$ . Thus we may assume that $T_X$ embeds in the sublattice $U^{\oplus 3}$ of $\mathbb{L}_{K3}$. Thus $S_X = (T_X)^{\perp}_{\mathbb{L}_{K3}}$ contains primitively embedded $E_8 \oplus E_8$.

(iii) $\Rightarrow$ (iv) This follows from some known result of V. Nikulin [8]. Suppose $G$ is a cyclic subgroup of order 2 of $\mathrm{O}(H^2(X,\mathbb{Z})$ and let $S_G = (H^2(X,\mathbb{Z})^G)^\perp$ be contained in $S_X$, negative definite and has no elements of norm $-2$. Then there exists an involution $\sigma$ of $X$ with 8 isolated fixed points such that $(\sigma^*) = G$. To apply Nikulin's theorem to our situation we consider the sublattice of $E_8 \oplus E_8$ of elements $(x,-x)$. It is isomorphic to $E_8(2)$ (i.e. $E_8$ with the quadratic form multiplied by 2). It is a negative definite lattice with no elements of norm $-2$. One can define an involution $\iota$ of $\mathbb{L}_{K3}$ such that $E_8(2)$ is contained in $S_X \cap (H^2(X,\mathbb{Z})^{(\iota)})^\perp$. By Nikulin's Theorem, there exists an automorphism $\sigma$ of $X$ such that $\sigma^* = \iota$ and $S_{(\sigma)} = E_8(2)$. We have $X/(\sigma)$ has 8 ordinary nodes, and its minimal resolution is a K3-surface $Y$. The orthogonal complement of $S_{(\sigma)}$ in $S_X$ contains a sublattice of $E_8 \oplus E_8$ of elements $(x,x)$. Its image in $S_Y$ is a sublattice isomorphic to $E_8$ and its orthogonal complement in $S_Y$ contains the primitive sublattice $N$ generated by the classes of the exceptional curves and one half if their sum. Thus $S_Y$ contains the sublattice $E_8 \oplus N$ of rank 16. One can show that a K3 surface containing such a lattice is birationally isomorphic to a Kummer surface $\mathrm{Kum}(A)$ of some abelian surface. It is another well-known theorem of Nikulin. The lattice $E_8 \oplus N$ is generated by 16 exceptional curves of its minimal resolution and the one-half of their sum.

(iv) $\Rightarrow$ (i) The involution $\sigma$ acts identically on $H^0(X,\Omega^2_X)$, and hence acts identically on $T_X$. This implies that $T_Y = \pi_*(T_X)$, where $\pi : X \dashrightarrow Y$ is the rational projection map. Also we know that $q_*(T_A) = T_Y$, where $q : A \to \mathrm{Kum}(A)$. This implies that $T_Y = T_X(2) = T_A(2)$ and hence $T_X \cong T_A$. One can also show that this isometry of lattices is a Hodge isometry.

We apply this to our situation. Recall that the Picard lattice $S_X$ of any $\check{M}_n$-polarized K3 surface $X$ contains the sublattice isomorphic to $E_8 \oplus E_8$. Assume that rank $S_X = 19$ so that $T_X = U \oplus U \oplus \langle -2n\rangle$. Let $\sigma$ be the corresponding Nikulin involution. Then a minimal resolution $Y$ of the quotient $Y' = X/(\sigma)$ is a K3 surface with $T_Y = T_X(2)$. I claim that $Y = \mathrm{Kum}(E \times E')$, where $E$ is an elliptic curve and $E' = E/(\lambda)$ for a subgroup $\lambda$ of order $n$. The pair $(E,\lambda)$ represents the corresponding point on $X_0(n)^+$. The Picard lattice of $E \times E'$ is easy to find. It is generated the numerical divisor classes $(f,f',g)$ of $E \times \{0\}, \{0\} \times E'$ and the graph of the map $E \to E' = E/(\lambda)$. The quadratic form is defined by the matrix $\begin{pmatrix} 0 & 1 & n \\ 1 & 0 & 1 \\ n & 1 & 0 \end{pmatrix}$. Its discriminant group is $\mathbb{Z}/n2\mathbb{Z}$. One can show that the isomorphism class of an even lattice of signature $(1,2)$ is uniquely determined by its discriminant group, if the latter is cyclic. Thus $S_{E\times E'} \cong U\otimes\langle 2n\rangle$ and $T_{E\times E'} \cong U\oplus U\oplus\langle -2n\rangle$. By Kondō's Lemma from [6], the fact that $S_X$ contains $U\oplus E_8\oplus E_8$ implies that there exists an elliptic fibration $f: X\to \mathbb{P}^1$ with a section and two singular fibers of type $\tilde{E}_8$. The Nikulin involution $\sigma$ acts on $X$ preserving this fibration and interchanging the two fibers. The quotient admits an elliptic fibration with a singular fiber of type $\tilde{E}_8$. The involution acts with two fixed points on the base of the fibration. Since $\sigma$ has 8 fixed points, the two fibers are nonsingular, and $\sigma$ has 4 fixed points on each fiber. The images of these fibers on the quotient surface are two fibers of type $\tilde{D}_4 = I_0^*$. The cover $X \dashrightarrow Y$ is defined by the double cover ramified over eight reduced components of these two fibers.

One can show (see [13]) that an elliptic surface with two fibers of type $\tilde{E}_8$ can be given by the Weierstrass equation

$$y^2 = x^3 - 3\alpha t_0^4t_1^4x + t_0^5t_1^5(t_0^2+t_1^2-2\alpha t_0t_1) = x^3 + A(t_0,t_1)x + B(t_0,t_1),$$

for some constants $\alpha,\beta$. The discriminant of the right-hand side cubic polynomial is equal to

$$\Delta = 4A^3+27B^2 = 27t_0^{10}t_1^{10}[4(\beta^2-\alpha^3)t_0^2t_1^2 + (t_0^2+t_1^2)(t_0^2+t_1^2-4\beta t_0t_1)].$$

The two fibers of type are over the point $(t_0:t_1) = (0:1)$ and $(1:0)$.

The Kummer surface with an elliptic pencil with fibers of types $\tilde{E}_8, \tilde{D}_4, \tilde{D}_4$ has the Weierstrass equation

$$y^2 = x^3 - 3\alpha u_0^4(u_1^2-4u_0^2)^2x + u_0(u_1-2\beta u_0)(u_1^2-4u_0^2)^3.$$

The discriminant is equal to

$$\Delta = 27u_0 10(u_1^2-4u_0^2)^6(4(\beta^2-\alpha^3)u_0^2 - 4\beta u_0u_1 + u_1^2).$$

The singular fibers are over the point $(u_0:u_1) = (0:1),(1:2),(1:-2)$. Choose two complex numbers $j_1,j_2$ such that

$$j_1j_2 = \alpha^3, \quad j_1+j_2 = 1+\alpha^3-\beta^2.$$

Then the Kummer surface is isomorphic to the Kummer surface of $E_{j_1}\times E_{j_2}$, where the subscript indicates the absolute invariant of the elliptic curve. In our case $j_1 = j(\tau), \tau\in\mathbb{H}$, hence $j_2 = j(n\tau)$. The numbers $(j_1,j_2)$ satisfy the *modular equation* of degree $\mu_n = n\prod_{p|n}(1+\frac{1}{p})$. It is an equation $f_n(x,y) = 0$ with integer coefficients. For example,

$$f_2(x,y) = (x-y^2)(x^2-y) + 2^4\cdot 3\cdot 31xy(x+y) - 2^4 3^4 5^3(x^2+y^2)+$$
$$2^8\cdot 7\cdot 61\cdot 373xy + 2^8 3^7\cdot 5^6(x+y) - 2^{12}3^9 5^9 = 0,$$

and

$$f_3(x,y) = x(x+2^{15}\cdot 3\cdot 5^3)^3 + y(y+2^7\cdot 3\cdot 5^3)^3 - x^3y^3+$$
$$2^3\cdot 3^2\cdot 31x^2y^2(x+y) - 2^2\cdot 3^3\cdot 9907xy(x^2+y^2) + 2\cdot 3^4\cdot 13\cdot 193\cdot 6367x^2y^2+$$
$$2^{16}\cdot 3^5\cdot 5^3\cdot 17\cdot 263xy(x+y) - 2^{31}\cdot 5^6\cdot 22973xy = 0.$$

Now the involution $\sigma$ of our K3 surface $X$ is defined by the formula

$$(x, y, t_0, t_1) \mapsto (x, -y, t_1, t_0).$$

The rational map $X \to \mathrm{Kum}(E \times E')$ is defined by the formula

$$(x, y, t = t_1/t_0) \mapsto (x(t - t^{-1})^2, y(t - t^{-1})^3, t + t^{-1}).$$

Note that the surface $X$ is sandwiched between the Kummer surface, i.e. there exists an involution $\tau$ on the Kummer surface such that the quotient is birationally isomorphic to $X$. The involution $\tau$ is defined by

$$\tau : (x, y, t_0, t_1) \mapsto (x, y, t_0^2, t_1^2).$$

Now let us do all of this over the moduli space of K3 surfaces $\mathcal{M}_{K3,\check{M}_n}$ and the moduli space $X_0(n)^+$ of pairs of isogenous elliptic curves. The modular curve $X_0(n)$ is the coarse moduli space of pairs $(E, \lambda \subset E[n])$. Unfortunately, because $\Gamma_0(n)$ contains the center of $\mathrm{SL}_2(\mathbb{Z})$, it is not the fine moduli space, i.e. there is no universal family over $X_0(n)$[10]. The subgroup $\mathbb{Z}^2 \rtimes \{\pm I_2\}$ of $\Gamma_0(n)$ acts on $\mathbb{C} \times \mathbb{H}$ by $(z, \tau) \mapsto (\pm z + m\tau + n, \tau)$. The quotient is isomorphic to universal Kummer curve of an elliptic curve together with a subgroup of order $n$. After we minimally resolve the singularities of the quotient, we obtain a ruled surface $Z_0(n) \to X_0(n)$. If $n$ is odd (resp. even) it comes with $\frac{n+1}{2}$ (resp. $\frac{n+2}{2}$) sections. The singular fibers of $Z_0(n) \to X_0(n)$ lie over the $\Gamma_0(n)/\{\pm I_2\}$-orbits of points in $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \infty$ with non-trivial stabilizer subgroups. Let $r_2, r_3$ denote the number of orbits of points in $\mathbb{H}$ with stabilizers of order 2,3, and $r_\infty$ be the number of cusps, the orbits of points on the boundary $\mathbb{H}^* \setminus \mathbb{H}$. We have (see [12]):

$$\begin{aligned}
r_2 &= \begin{cases} 0 & \text{if } 4|N, \\ \prod_{p|N}(1 + (\frac{-1}{p})) & \text{otherwise.} \end{cases} \\
r_3 &= \begin{cases} 0 & \text{if } 9|N, \\ \prod_{p|N}(1 + (\frac{-3}{p})) & \text{otherwise.} \end{cases} \\
r_\infty &= \sum_{d|N, d>0} \phi((d, \frac{N}{d})).
\end{aligned}$$

Here $\phi$ is the Euler function and $(\frac{-}{p})$ is the Legendre symbol of quadratic residue. We have

$$(\frac{-1}{p}) = \begin{cases} 0 & \text{if } p = 2, \\ 1 & \text{if } p \equiv 1 \bmod 4, \\ -1 & \text{if } p \equiv 3 \bmod 4, \end{cases}$$

$$(\frac{-3}{p}) = \begin{cases} 0 & \text{if } p = 3, \\ 1 & \text{if } p \equiv 1 \bmod 3, \\ -1 & \text{if } p \equiv 2 \bmod 3. \end{cases}$$

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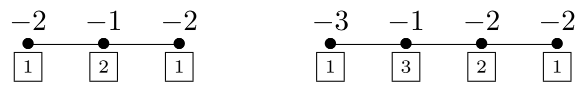
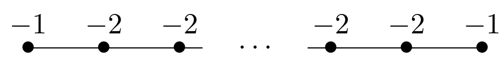
[10]However, the curves $X_1(n)$ and $X(n)$ are fine moduli spaces for $n \geq 3$. The fine moduli space is the modular elliptic surfaces $\pi : S_1(n) \to X_1(n)$ and $S(n) \to X(n)$

The fiber of points with stabilizer of order 2 (resp. 3) is the union of curves with dual graph pictured in the following diagrams, where the numbers indicate the self-intersection of the irreducible components and the boxed numbers indicate the multiplicity of the component:




order 2 order 3

We say that fibers over points with stabilizer of order 2 (resp. 3) is of type $A$ (resp B). For any cusp $c$ let $b$ denote its *width*. We have the following diagrams for fibers over cusps with width $b \geq 2$:




The number of components with self-intersection $-2$ is equal to $\frac{b-2}{2}$ if $b$ is even and $\frac{b-3}{2}$ of $b$ is odd. If $b = 1$, the fiber is nonsingular. We say that the fiber is of type $C_b$.

For example, take $n = 3$. We have $r_2 = 0, r_3 = 1$. There are two cusps with width 3 and 1. Thus $Z_0(3) \to \mathbb{P}^1$ is a ruled surface with one singular fiber of type $B$ and one singular fiber of type $T_3$.

To construct the latter we consider the fiber product $Z \to X_0(n)$ of the ruled surfaces $\pi : Z_0(n) \to X_0(n)$ and $\pi' := F \circ \pi : Z_0(n) \to \mathbb{P}^1 \to \mathbb{P}^1$, where $F$ is the Fricke involution. Let $S$ be the set of fixed points of the Fricke involution. It is known that $S$ does not contain cusps. A fixed point of $F$ corresponds to an elliptic curve $E$ and a subgroup of order $n$ such that $E \to E/\lambda$ is an isomorphism. Thus $E \to E/\lambda$ defines an isogeny of $E$. Since the composition $E \to E/\lambda \to E = E/E[n]$ is equal to multiplication by $n$, we see that $\lambda$ belongs to a ring $\mathfrak{o}$ of complex multiplications of $E$. It is a subring of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-n})$. As is well-known elements of $\mathfrak{o}$ can be written in the form $a + fb\frac{1+\sqrt{-d}}{2}$ (resp. $a + fb\sqrt{-d}$) for some integers $a, b$ and fixed positive integer $f$ if $n \equiv 1 \mod 4$ (resp. if $n \equiv 2, 3 \mod 4$). The isogeny $E \to E/\lambda$ corresponds to an element $\alpha$ of $\mathfrak{o}$ such that $\alpha^2 = -n$. If $n \equiv 1 \mod 4$, then an easy computation shows that $\alpha = \pm\sqrt{-d}$ and $fb = 2$ (resp. $fb = 1$).

Then we take the fibered product $\Phi : S_0(n) \times_{\mathbb{P}^1} S_0(n)$ and its quotient by the lift of the Fricke involution $F$ acting on the base and inducing the natural isomorphism $\Phi^{-1}(x) = \pi^{-1}(x) \times \pi'^{-1}(F(x)) \to \Phi^{-1}(F(x)) = \pi^{-1}(F(x)) \times \pi'^{-1}(x) = E' \times E$.

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