Higgs bundles and mirror symmetry 3

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REMARKS

SYZ MIRROR SYMMETRY

- Calabi-Yau manifold M^n: ω symplectic form,

  Ω = real part of a holomorphic n-form

- special Lagrangian fibration: $p : M \to B$

  (ω, Ω vanish on fibres)

- fibres are tori T_b

- mirror = dual fibration, fibre over b = moduli space of flat $U(1)$-bundles over T_b

- fibres of $p : M \to B$ are tori
- flat tori – linear vector fields
- fibres of mirror are abelian **groups**
- lack of symmetry $\sim$ gerbes

- mirror of $Sp(m)$ moduli space $=$ $SO(2m+1)$ moduli space

- two components: spin/non-spin

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- Lagrangian L, $L \cap A =$ union of translates of B

- mirror B^0 connected

CONCLUSIONS FROM LAST LECTURE

- “most” $\mathbf{C}^*$-invariant Lagrangians meet a smooth fibre in dimension zero

  $\Rightarrow$ support of mirror is whole moduli space

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- $\Rightarrow$ switch attention to hyperholomorphic bundles

REAL FORMS

- complex structure I: moduli space of (stable) pairs (A, Φ)

  $G = U(n)$ vector bundle V, $\Phi \in H^0(\Sigma, \operatorname{End} V \otimes K)$

- complex structure J: flat G^c-connection

  $\nabla_A + \Phi + \Phi^*$ (representations $\pi_1(\Sigma) \to G^c$)

- complex structure K: flat G^c-connection

  $\nabla_A + i\Phi - i\Phi^*$

REAL FORM G^r

- $K \subset G^r$ maximal compact
- principal K^c-bundle
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$
- Higgs field $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$
- holonomy of $\nabla + \Phi + \Phi^* \in G^r$

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$\mathbf{C}^*$-invariant

- Higgs field $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$
- holonomy of $\nabla + \Phi + \Phi^* \in G^r$

- moduli space of flat G^r-connections: $\mathrm{Hom}(\pi_1, G^r)/G^r$

- fixed point set of involution on $\mathcal{M}$

- I-holomorphic, J, K-antiholomorphic

- BAA-brane

- $0 \to H^1(\Sigma, \mathbf{R}) \to H^1(\Sigma, \mathbf{R}^*) \to \mathbf{Z}_2^{2g} \to 0$

- $\alpha^{10} + \overline{\alpha^{10}} \in H^1(\Sigma, \mathbf{R})$

- each component I-holomorphically parametrized

  by $\alpha^{10} \in H^0(\Sigma, K) \cong \mathbf{C}^g$

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- $H^1(\mathbf{\Sigma}, \mathbf{R}^*) = 2^{2g}$ *holomorphic* sections of p

  $H^1(\mathbf{\Sigma}, \mathbf{R}^*) =$ *real* points of $(\mathbf{C}^*)^{2g}$

- L = moduli space of flat G^r-connections

- G^r = split real form e.g. $SL(n, \mathbf{R}), Sp(2m, \mathbf{R}), \ldots$

  $\Rightarrow L \cap A$ = 2-torsion points

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- $L =$ moduli space of flat G^r-connections

- for many G^r, L does not intersect the smooth fibres

- ... but "many" hyperkähler submanifolds do not intersect the smooth fibres

$U(m,m) \subset GL(2m, \mathbf{C})$

- maximal compact $U(m) \times U(m)$
- bundle $V = V_+ \oplus V_-$ Higgs field $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$
- characteristic class $c_1(V_+) \in H^2(\Sigma, \mathbf{Z})$
- $\Rightarrow$ different topological components

L.Schaposnik, *Spectral data for $U(m,m)$ Higgs bundles*, IMRN, **11** (2015) 3486 – 3498.

- spectral curve $\det(x - \Phi) = x^{2m} + a_2 x^{2m-2} + \ldots + a_{2m}$

- involution $\sigma(x) = -x$ on S

- $V = \pi_*(U\pi^* K^{(2m-1)/2})$, $U \in \mathrm{Jac}(S)$

- line bundle $U \in \mathrm{Jac}(S)$, $\sigma^* U \cong U$

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- fixed points $a_{2m} = 0$ $4m(g-1)$ points
- $\sigma^* U \cong U$ action at fixed points ± 1
- action $+1$ everywhere $\Rightarrow U$ pulled back from $\bar{S} = S/\sigma$

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- $L \cap A = 2^{4m(g-1)-1}$ copies of $\mathrm{Jac}(\bar{S})$

- $(L \cap A)^0 \cong \mathsf{P}(S, \bar{S})$

- mirror supported on the family of Prym varieties over $H^0(\Sigma, K^2) \oplus H^0(\Sigma, K^4) \oplus \cdots \oplus H^0(\Sigma, K^{2m})$

- $= Sp(m)$ moduli space in $U(2m)$ moduli space

- ... which is hyperkähler.

S-Duality Of Boundary Conditions in $\mathcal{N} = 4$ Super Yang-Mills Theory

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Abstract

By analyzing brane configurations in detail, and extracting general lessons, we develop methods for analyzing S-duality of supersymmetric boundary conditions in $\mathcal{N} = 4$ super Yang-Mills theory. In the process, we find that S-duality of boundary conditions is closely related to mirror symmetry of three-dimensional gauge theories, and we analyze the IR behavior of large classes of quiver gauge theories.

Table 3: The first column lists the unbroken subgroups H in boundary conditions in $SU(n)$ gauge theory that are defined by an involution τ. The second column lists the unbroken gauge symmetry $\widetilde{H}$ of the S-dual boundary condition. The third column describes the Nahm pole, if any, that is part of the reduction of the dual gauge group from $SU(n)$ to $\widetilde{H}$. The fourth column describes the matter system that is coupled to $\widetilde{H}$. (The hypermultiplets indicated are in the fundamental representation of $Sp(n)$.)

H	$\widetilde{H}$	Nahm Pole	Matter System
$SO(n)$	$SU(n)$	None	Non-trivial SCFT
$Sp(n)$	$SU(n/2)_2$	$n = 2 + 2 + \cdots + 2$	None
$S(U(n/2) \times U(n/2))$	$Sp(n)$	None	Hypermultiplets
$S(U(p) \times U(q)),\ p > q$	$Sp(2q)$	$n = (p - q) + 1 + 1 + \cdots + 1$	None

real forms of G^c

complex subgroups of ${}^L G^c$

	$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{g}$	$\check{\mathfrak{g}}$	$\check{\mathfrak{h}}$	Remarks
AI	$\mathfrak{sl}_n(\mathbb{R})$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	split
AII	$\mathfrak{su}^*(2n)$	$\mathfrak{sl}_{2n}(\mathbb{C})$	$\mathfrak{sl}_{2n}(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	
AIII/AIV	$\mathfrak{su}(p,q)$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sp}_p(\mathbb{C})$	$p \leq q$
					$p+q=n$
					quasi-split if $q=p$
					or $q=p+1$
BI/BII	$\mathfrak{so}(p,q)$	$\mathfrak{so}_{2n+1}(\mathbb{C})$	$\mathfrak{sp}_n(\mathbb{C})$	$\mathfrak{sp}_p(\mathbb{C})$	$p<q$
					$p+q=2n+1$
					split if $q=p+1$
CI	$\mathfrak{sp}_n(\mathbb{R})$	$\mathfrak{sp}_n(\mathbb{C})$	$\mathfrak{so}_{2n+1}(\mathbb{C})$	$\mathfrak{so}_{2n+1}(\mathbb{C})$	split
CII	$\mathfrak{sp}(p,q)$	$\mathfrak{sp}_n(\mathbb{C})$	$\mathfrak{so}_{2n+1}(\mathbb{C})$	$\mathfrak{sp}_p(\mathbb{C})$	$p \leq q$
					$p+q=n$
DI/DII	$\mathfrak{so}(n,n)$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{so}_{2n}(\mathbb{C})$	split
	$\mathfrak{so}(p,q)$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{so}_{2p+1}(\mathbb{C})$	$p<q$
					$p+q=2n$
					quasi-split if $q=p+2$
DIII	$\mathfrak{so}^*(2n)$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{sp}_p(\mathbb{C})$	$p=[n/2]$
EI	$\mathfrak{e}_{6(6)}$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	split
EII	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	quasi-split
EIII	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{so}_5(\mathbb{C})$	
EIV	$\mathfrak{e}_{6(-26)}$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{sl}_3(\mathbb{C})$	
EV	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_7(\mathbb{C})$	split
EVI	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	
EVII	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{sp}_3(\mathbb{C})$	
EVIII	$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_8(\mathbb{C})$	$\mathfrak{e}_8(\mathbb{C})$	$\mathfrak{e}_8(\mathbb{C})$	split
EIX	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_8(\mathbb{C})$	$\mathfrak{e}_8(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	
FI	$\mathfrak{f}_{4(4)}$	$\mathfrak{f}_4(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	split
FII	$\mathfrak{f}_{4(-20)}$	$\mathfrak{f}_4(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	$\mathfrak{sl}_2(\mathbb{C})$	
G	$\mathfrak{g}_{2(2)}$	$\mathfrak{g}_2(\mathbb{C})$	$\mathfrak{g}_2(\mathbb{C})$	$\mathfrak{g}_2(\mathbb{C})$	split

TABLE 1. Associated Lie algebras $\check{\mathfrak{h}}$ for non-compact real Lie algebras $\mathfrak{g}_{\mathbb{R}}$ with simple complexifications $\mathfrak{g}$. Notation following É. Cartan, and [Hel78].

THE NADLER GROUP

D.Nadler, *Perverse sheaves on real loop Grassmannians,* Invent. Math. **159** (2005) 1–73

- $G^r \subset G^c$
- $\Rightarrow \widehat{H}^c \subset {}^L G^c$

$U(m,m) \subset GL(2m, \mathbf{C})$

- maximal compact $U(m) \times U(m)$
- bundle $V = V_+ \oplus V_-$ Higgs field $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$
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L.Schaposnik, *Spectral data for* $U(m,m)$ *Higgs bundles*, IMRN, **11** (2015) 3486 – 3498.

- Lagrangians $L_0, L_1, \ldots$

- same support of the BBB-brane $\Rightarrow$

  they must differ through the hyperholomorphic vector bundle

HYPERHOLOMORPHIC BUNDLES

- connection with curvature of type $(1,1)$ wrt I, J, K
- 4 dimensions = anti-self-dual
- $\Leftrightarrow$ holomorphic bundle on twistor space

- Levi-Civita connection is hyperholomorphic

- Higgs bundle tangent space $(\dot{A},\dot{\Phi})\in\Omega^{01}(\mathfrak{g})\oplus\Omega^{10}(\mathfrak{g})$

- $\bar{\partial}_A\dot{\Phi}+[\dot{A},\Phi]=0$ modulo $(\dot{A},\dot{\Phi})=(\bar{\partial}_A\psi,[\psi,\Phi])$

- elliptic complex

  $0\rightarrow\Omega^{00}(\mathfrak{g})\rightarrow\Omega^{01}(\mathfrak{g})\oplus\Omega^{10}(\mathfrak{g})\rightarrow\Omega^{11}(\mathfrak{g})\rightarrow 0$

- tangent space to $\mathcal{M}$ = first cohomology group

- Dolbeault version of hypercohomology
- sequence of sheaves $\mathcal{O}(\mathfrak{g}) \stackrel{\mathrm{ad}\,\Phi}{\to} \mathcal{O}(\mathfrak{g} \otimes K)$
- tangent space to $\mathcal{M}$ = first hypercohomology group $\mathbb{H}^1$
- varies holomorphically over $\mathcal{M}$ with complex structure I

- Higgs bundle equations $F_A + [\Phi, \Phi^*] = 0 \Rightarrow$ flat connection

- variation: $d_A(\dot{A} + \dot{\Phi} + \dot{\Phi}^*) + [\Phi + \Phi^*, \dot{A} + \dot{\Phi} + \dot{\Phi}^*] = 0$

- tangent space to $\mathcal{M}$ = first de Rham cohomology group H^1 of flat connection

- varies holomorphically over $\mathcal{M}$ with complex structure J

- Hodge theory for elliptic complex

  $0 \to E_0 \stackrel{d}{\to} E_1 \stackrel{d}{\to} E_2 \to 0$

- $d + d^* : E_0 \oplus E_2 \to E_1$

- same operator for each complex −"Dirac" operator $\mathbf{D}$

- coker $\mathbf{D}$ defines a hyperholomorphic bundle over $\mathcal{M}$

- replace $\mathfrak{g}$ by any representation of G

- hypercohomology of sequence of sheaves: $\mathcal{O}(V) \stackrel{\Phi}{\to} \mathcal{O}(V \otimes K)$

- coker $\mathbf{D}$ defines a hyperholomorphic bundle over $\mathcal{M}$

- "Dirac-Higgs bundle" (if a universal bundle over $\mathcal{M} \times \Sigma$ exists)

VECTOR REPRESENTATION

- $\mathcal{O}(V) \xrightarrow{\Phi} \mathcal{O}(V \otimes K)$

- $0 \to H^1(\ker \Phi) \to \mathbb{H}^1 \to H^0(\mathrm{coker}\, \Phi) \to 0$

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- open covering $U_\alpha, \ldots$

  $\theta_{\alpha\beta}$ holomorphic section of V on $U_\alpha \cap U_\beta$

  ψ_α on U_α

- $\Phi\theta_{\alpha\beta} = \psi_\beta - \psi_\alpha \Rightarrow$ class in $\mathbb{H}^1$

- project to cokernel $\Rightarrow \bar{\psi}_\beta = \bar{\psi}_\alpha$

VECTOR REPRESENTATION

- $\mathcal{O}(V) \stackrel{\Phi}{\to} \mathcal{O}(V \otimes K)$

- $0 \to H^1(\ker \Phi) \to \mathbb{H}^1 \to H^0(\operatorname{coker} \Phi) \to 0$

- $\det \Phi = 0$ on $x = 0$ and $\operatorname{coker} \Phi \cong L$

  so $\mathbb{H}^1 \cong \bigoplus_{x_i \in S \cap \{x=0\}} L_{x_i}$

- Dirac-Higgs bundle $\mathbf{V}$ hyperholomorphic

MIRROR SYMMETRY

- Lagrangian $L \subset \mathcal{M}$

- $E \in (L \cap A)^0 \subset \mathcal{M}^\vee$ line bundle on A trivial on $L \cap A$

  $H^0(L \cap A, E)$: basis vector for each component of $L \cap A$

- $E \in \mathcal{M}^\vee$ regular $\Rightarrow$ vector space $H^0(L \cap A, E)$

- universal bundle on family $A \times A^\vee \Rightarrow$ vector bundle on $\mathcal{M}^\vee$

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is this hyperholomorphic?

REAL FORM $U(m,m)$

- $L \cap A = 2^{4m(g-1)-1}$ copies of $\mathrm{Jac}(\bar{S})$

- and $\mathcal{M}^{\vee} = Sp(m)$-moduli space

- $\dim H^0(L \cap A, E) = 2^{4m(g-1)-1}$

- L has different topological components

  $\Rightarrow$ hyperholomorphic subbundles

- $\sigma^* U \cong U$
- action at fixed point set ± 1
- $c_1(V_+) \sim$ number of $+1$s
- basis vectors for $H^0(L \cap A, E) \sim$

  even subsets of $4m(g-1)$ zeros of a_{2m}

- $\mathcal{M}^\vee = Sp(m)$ moduli space

- $E \in A^\vee = \mathsf{P}(S, \bar{S})$

- $\{x_1, \ldots, x_\ell\} \subset S \cap \{x = 0\}$ defines

  $E_{x_1} \otimes E_{x_2} \otimes \cdots \otimes E_{x_\ell}$

- vector space $\displaystyle\bigoplus_{\{x_1,\ldots,x_\ell\} \subset S \cap \{x=0\}} E_{x_1} \otimes E_{x_2} \otimes \cdots \otimes E_{x_\ell}$

- Dirac-Higgs bundle $\mathbf{V} = \bigoplus_{x_\ell \in S \cap \{x=0\}} E_{x_\ell}$

- $\Lambda^\ell \mathbf{V} = \bigoplus_{\{x_1,\ldots,x_\ell\} \subset S \cap \{x=0\}} E_{x_1} \otimes E_{x_2} \otimes \cdots \otimes E_{x_\ell}$

- sum over ℓ-element subsets

  induced hyperholomorphic connection

- no universal bundle for $Sp(m)$

- local ones differ by a line bundle $L_{\alpha\beta}$ on

  $U_\alpha \cap U_\beta \subset \mathcal{M}^\vee$ of order 2

- ℓ even $\Rightarrow \Lambda^\ell \mathbf{V}_\alpha = \Lambda^\ell \mathbf{V}_\beta$ well-defined

- $SU(2)$ bundle V, S^3V symplectic

  quadratic moment map $\mu : S^3V \to \mathfrak{g}$

- $\psi \in H^0(\Sigma, S^3V \otimes K^{1/2})$

- $\{(V, \Phi) : \Phi = \mu(\psi)\}$ is Lagrangian

- $L \cap A =$ 3-torsion points : mirror?

NJH, *Spinors,Lagrangians and rank 2 Higgs bundles*, Proc LMS, **115** (2017) 33–54.

REVERSING THE MIRROR

- E.Franco & M.Jardim *Mirror symmetry for Nahm branes*, arXiv 1709.01314

- tensor product $(V_1 \otimes V_2, \Phi_1 \otimes 1 + 1 \otimes \Phi_2)$

- fix V_2, HK map $\mathcal{M}(U(m)) \to \mathcal{M}(U(mn))$

- pull back Dirac-Higgs

- ... Fourier-Mukai mirror

  supported on a Lagrangian L with $L \cap A$ finite

- $m = 1$ = Nahm transform

  J.Bonsdorff, *A Fourier transform for Higgs bundles*, Crelle **591** (2006) 21–48

- :a fixed Higgs bundle defines a hyperholomorphic bundle on $T^*\mathrm{Jac}(\Sigma)$

HYPERKÄHLER SUBMANIFOLDS

- “most” $\mathbf{C}^*$-invariant Lagrangians meet a smooth fibre A in a finite no of points

- ... can a hyperkähler submanifold?

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- ... can a hyperkähler submanifold?

- Not mirrors of $\mathbf{C}^*$-invariant Lagrangians

- ... which were subintegrable systems

- $\mathbf{H}^2 = \mathbf{C}^2 \oplus j\mathbf{C}^2$ projection $p(z,w) = w$

  $I(z,w) = (iz, -iw),\ J(z,w) = (-w, z)$

- $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad S = \{(z, -A\bar{z}) : z \in \mathbf{C}^2\}$

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- $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad S = \{(z, -A\bar{z}) : z \in \mathbf{C}^2\}$

- $S \cap p^{-1}(w) = (A\bar{w}, w)$ single point

- $I(z, -A\bar{z}) = (iz, -A\overline{(iz)})$

  $J(z, -A\bar{z}) = (A\bar{z}, z) = (w, -A\bar{w})$

SEMIFLAT METRIC

$$\omega_2 = \sum \frac{\partial^2 \phi}{\partial x_j \partial x_k} dx_j \wedge dy_k$$

$$\omega_1 + i\omega_3 = \frac{1}{2} \sum \omega_{jk} d(x_j + iy_j) \wedge d(x_k + iy_k)$$

- $S \subset \mathcal{M}$ hyperkähler submanifold

- restrict $x_1, \ldots, x_n$ Hamiltonian functions

  ... no longer Poisson-commute in general

- tangential components on S are the Hamiltonian vector fields Y_i of x_i restricted to S

- circle action: vector field X

- moment map for $\omega_1 = \phi$, also J-Kähler potential

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- in the semiflat metric X is horizontal $\Rightarrow JX, KX$ vertical

- S hyperkähler $\Rightarrow \dim S \cap A \geq 1$ if S is $\mathbf{C}^*$-invariant

- In general....

- JX, KX are linear vector fields tangent to $M \cap A$

- ... closure of an orbit $\Rightarrow$ abelian subvariety C

- $C^0 \Rightarrow \mathbf{C}^*$-invariant Lagrangian