Higgs bundles and mirror symmetry 2

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DUALITY

- Higgs bundle fibration $p : \mathcal{M} \to \mathcal{B}$

- generic fibre abelian variety A

- $=$ complex torus $+$ positive line bundle H

- SYZ mirror symmetry $\sim$ replace A by its dual $A^\vee$

$=$ moduli space of degree zero holomorphic line bundles on A

- $x \in A$, $L_x(y) = x + y$
- translation action $L_x : A \to A$
- $L_x^* H \otimes H^{-1}$ degree zero line bundle
- $A \mapsto A^\vee$ surjective, finite kernel
- $A = \mathrm{Jac}(S)$ isomorphism (A principally polarized)

- spectral curve $\pi : S \to \Sigma$

- $\deg \pi_* L = \deg V = 0$ if $L = U \otimes \pi^* K^{(n-1)/2}$, $\deg U = 0$

- $\Lambda^n V$ trivial if $\mathrm{Nm}(U) = 0$

- $\mathrm{Jac}(S) \sim$ linear equivalence of divisors

  $\mathrm{Nm}(x_1 + \ldots + x_k) = \pi(x_1) + \ldots + \pi(x_k)$

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- $\ker \mathrm{Nm} \stackrel{\mathrm{defn}}{=} \mathrm{P}(S, \Sigma) =$ Prym variety

THE GROUP $SU(n)$

- $U \in \mathsf{P}(S,\Sigma) \Rightarrow \Lambda^n V$ trivial

- structure group $SU(n)$

- $\Phi \in H^0(\Sigma, \mathfrak{sl}(n) \otimes K) \Rightarrow a_1 = 0 \in H^0(\Sigma, K)$

- generic fibre for $SU(n)$ Higgs bundles $\cong$ Prym variety

- Nm : Jac(S) $\to$ Jac(Σ) is dual to π^* : Jac(Σ) $\to$ Jac(S)

- so P(S, Σ)$^\vee \cong$ Jac(S)/π^* Jac(Σ)

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- so $\mathsf{P}(S, \Sigma)^\vee \cong \mathsf{Jac}(S)/\pi^* \mathsf{Jac}(\Sigma)$

- $\mathsf{Jac}(S)/\pi^* \mathsf{Jac}(\Sigma) \;\; \cong \mathsf{P}(S, \Sigma)/(\mathsf{P}(S, \Sigma) \cap \pi^* \mathsf{Jac}(\Sigma))$

- $\mathsf{Nm}\, \pi^* x = nx \Rightarrow$

$$\mathsf{P}(S, \Sigma)^\vee \cong \mathsf{P}(S, \Sigma)/\pi^* H^1(\Sigma, \mathbf{Z}_n)$$

THE GROUP $Sp(m)$

- E rank $2m$ symplectic vector bundle

- $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K) = H^0(\Sigma, S^2E \otimes K)$

- eigenvalues $\pm\lambda_i$

- spectral curve $S \subset |K|$:

  $0 = \det(x - \Phi) = x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m}$

- involution $\sigma(x) = -x$

- $\pi : S \to \Sigma$
- $E = \pi_* L$
- $x : L \to L \otimes \pi^* K$ and $\Phi = \pi_* x$
- .... where $L = U \pi^* K^{m-1/2}$ and $\sigma^* U \cong U^*$

- $p : S \to \bar{S} = S/\sigma$

- $\sigma^* U \cong U^* \Leftrightarrow U \in \mathsf{P}(S, \bar{S})$

- abelian variety = Prym $\mathsf{P}(S, \bar{S})$

- dual $\mathsf{P}(S, \bar{S})^\vee \cong \mathsf{P}(S, \bar{S})/p^* H^1(\bar{S}, \mathbf{Z}_2)$

LANGLANDS DUALITY

Invent. math. 153, 197–229 (2003)
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Inventiones mathematicae

Mirror symmetry, Langlands duality, and the Hitchin system

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Abstract. Among the major mathematical approaches to mirror symmetry are those of Batyrev-Borisov and Strominger-Yau-Zaslow (SYZ). The first is explicit and amenable to computation but is not clearly related to the physical motivation; the second is the opposite. Furthermore, it is far from obvious that mirror partners in one sense will also be mirror partners in the other. This paper concerns a class of examples that can be shown to satisfy

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Electric-Magnetic Duality And The Geometric Langlands Program

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- $\mathcal{M}(G)$ Higgs bundle moduli space

- hyperkähler

- holomorphic Lagrangian fibration

- ....its mirror is $\mathcal{M}(^L G)$ where $^L G$ is the Langlands dual group

R.Donagi & T.Pantev, *Langlands duality for Hitchin systems,* Invent. math. **189** (2012), 653–735.

- G and ${}^LG$ are Langlands dual groups if..

- ... their root systems are dual

- roots $\leftrightarrow$ coroots, characters $\leftrightarrow$ 1-parameter subgroups

- ${}^LU(n) = U(n)$

  ${}^LSU(n) = PSU(n) = SU(n)/\mathbf{Z}_n$

  ${}^LSp(m) = SO(2m+1)$

- ${}^{L}U(n) = U(n)$: $\mathsf{Jac}(S)^{\vee} \cong \mathsf{Jac}(S)$

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- ${}^{L}SU(n) = PSU(n) = SU(n)/\mathbf{Z}_n$

- $\mathbf{Z}_n$ action $(V, \Phi) \mapsto (V \otimes L, \Phi)$ where L^n is trivial

  $\mathbf{Z}_n$ quotient of $SU(n)$-moduli space, $PSU(n)$-Higgs bundles

- $\mathsf{P}(S, \Sigma)^{\vee} \cong \mathsf{P}(S, \Sigma)/\pi^* H^1(\Sigma, \mathbf{Z}_n)$

$Sp(2m, \mathbf{C})$ AND $SO(2m+1, \mathbf{C})$

$Sp(2m, \mathbf{C})$

- E rank $2m$ symplectic vector bundle
- $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K) = H^0(\Sigma, S^2E \otimes K)$
- eigenvalues $\pm\lambda_i$
- spectral curve $S \subset |K|$:

  $0 = \det(x - \Phi) = x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m}$
- involution $\sigma(x) = -x$

$SO(2m+1, \mathbf{C})$

- V rank $(2m+1)$ orthogonal vector bundle

- $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K) = H^0(\Sigma, \Lambda^2 V \otimes K)$

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- V rank $(2m+1)$ orthogonal vector bundle
- $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K) = H^0(\Sigma, \Lambda^2 V \otimes K)$
- eigenvalues $0 \pm \lambda_i$
- kernel $\sim \Phi^m \in \Lambda^{2m} V \otimes K^m \cong V \otimes K^m$
- reducible spectral curve

  S

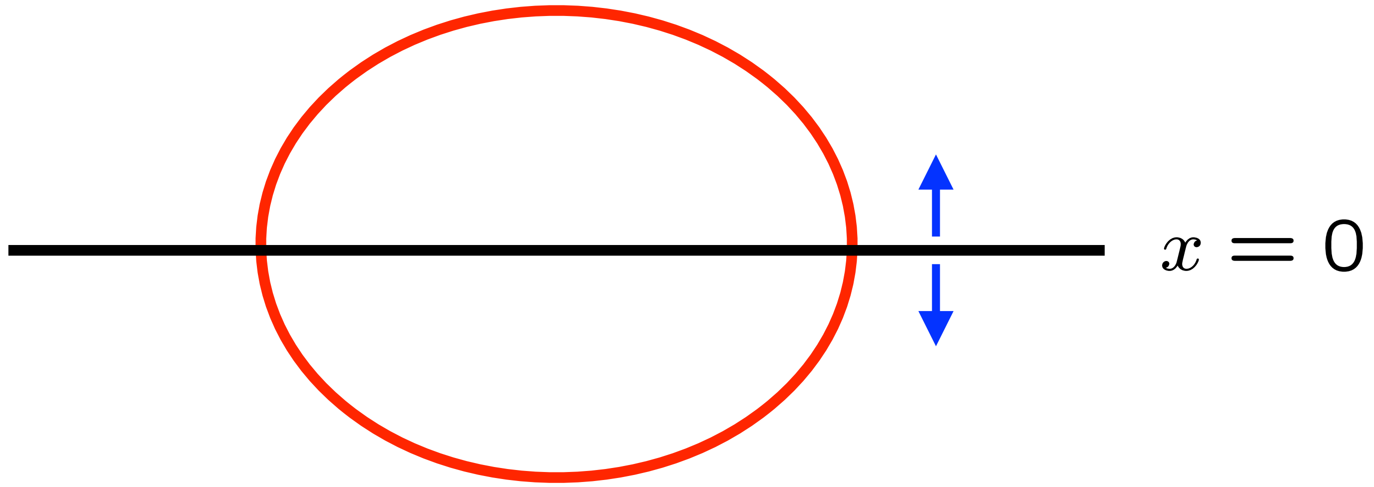
  $0 = \det(x - \Phi) = x(x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m})$

- $V = \pi_* L$ where ...

- on $x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m} = 0$

  $L = U\pi^* K^m$ and $U \in P(S, \bar{S})$

- on $x = 0 \cong \Sigma$, $L = K^m$



$x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m} = 0$

- $x = 0$ fixed point set of σ

- $\sigma^* U \cong U^* \Rightarrow$ trivialization of U^2 on $x = 0$

- $K^m \cong UK^m \Rightarrow \pm 1$ choice

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- $x = 0 \Leftrightarrow a_{2m}(z) = 0$: $4m(g-1)$ points

- $2^{4m(g-1)}$ covering of $P(S, \bar{S})$

- overall ± 1

  $+$

  $SO(2m+1)$-bundle spin/non-spin

- $2^{4m(g-1)-2}$ covering of $P(S, \bar{S})$

- $= P(S, \bar{S})/p^* H^1(\bar{S}, \mathbf{Z}_2)$

  $=$ dual of $P(S, \bar{S})$

BRANES

- symplectic geometry

  A-brane = Lagrangian submanifold + flat vector bundle

- holomorphic geometry

  B-brane = complex submanifold + holomorphic bundle

- ... or generalizations, sheaves etc.

SYZ MIRROR SYMMETRY

- Calabi-Yau manifold M^n: ω symplectic form,

  Ω = real part of a holomorphic n-form

- special Lagrangian fibration: $p : M \to B$

  (ω, Ω vanish on fibres)

- fibres are tori T_b

- mirror = dual fibration, fibre over b = moduli space of flat $U(1)$-bundles over T_b

- $L \subset T_x M$ Lagrangian subspace
- suppose $T_x L = V \oplus H \subset T_F M \oplus p^* TB$
- Lagrangian $\Rightarrow H = V^0$
- $V^0 \subset T_F^* \cong TB$ then $V^0 \oplus iV^0$ complex

- hyperkähler: complex structures I, J, K

- symplectic forms $\omega_1, \omega_2, \omega_3$

- BAA-brane = holomorphic Lagrangian submanifold wrt I + flat connection

- BBB-brane = hyperkähler submanifold + hyperholomorphic bundle

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mirror symmetry is supposed to interchange these

BBB-BRANES

- hyperkähler submanifold

- strong condition

- Kähler $\Rightarrow$ second fundamental form S complex linear

  $S(IX, JY) = IS(X, JY) = IJS(X, Y) = JIS(X, Y) = 0$

- $\Rightarrow$ totally geodesic

EXAMPLES

- points, whole manifold

- Higgs bundles for subgroup $H \subset G$

- pull-back of Higgs bundles from a map $f : \Sigma \to C$

- fixed point set of a triholomorphic automorphism of $\mathcal{M}$

  e.g.

  i) induced action of finite group holomorphic action on Σ

  ii) $V \mapsto V \otimes L$ where L^n trivial.

BAA BRANES

- $\mathcal{N} \subset \mathcal{M}$ = moduli space of stable bundles $(V, 0)$
- torus fibres of the integrable system
- $T^*\mathcal{N} \subset \mathcal{M}$ open embedding ....
- .... closure of conormal bundle of submanifold of $\mathcal{N}$

BAA BRANES

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- .... closure of conormal bundle of submanifold of $\mathcal{N}$

any

SYZ MIRROR SYMMETRY

- $p : \mathcal{M} \to \mathcal{B}$ integrable system

  $L \subset \mathcal{M}$ complex Lagrangian submanifold

- $p : L \to p(L)$ suppose generically $p(L) \subset \mathcal{B}^{\text{reg}}$

  $L \cap A \subset A$ compact subvariety of abelian variety

- **Define** $(L \cap A)^0 \subset A^\vee$ = line bundles on A trivial on $L \cap A$

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- .... = support of a BBB-brane?

EXAMPLE

- L = a single fibre A
- $p(L)$ a point $a \in \mathcal{B}$
- $(L \cap A) = A \Rightarrow (L \cap A)^0 = 0$
- $\Rightarrow$ BBB-brane is a single point

EXAMPLE

- $L =$ (closure of) a cotangent fibre of $T^*\mathcal{N}$
- $(L \cap A)$ finite set $\Rightarrow (L \cap A)^0 = A^\vee$
- $\Rightarrow$ BBB-brane is the whole manifold
- How general is this?

$\mathbb{C}^*$-INVARIANT LAGRANGIANS

- $\mathbf{C}^*$-action $(V, \Phi) \mapsto (V, \lambda\Phi)$

  generated by holomorphic vector field X

- on $T^*\mathcal{N}$ scales cotangent fibres

- in general preserves conormal bundles

- moves generic fibres ($0 \in \mathcal{B}$ only fixed point)

- Suppose L is a $\mathbf{C}^*$-invariant complex Lagrangian

- $i_X(\omega_2 + i\omega_3) = i_X\omega$ vanishes on L

- $i_X\omega$ nonzero on A

  (variation of Lagrangian A in the direction X)

- $i_X\omega$ holomorphic 1-form on A which vanishes on $L \cap A$

- Universal property: a map $M \to A$ factors through the Albanese variety $M \to \mathrm{Alb}(M) \to A$

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- Universal property: a map $M \to A$ factors through the Albanese variety $M \to \text{Alb}(M) \to A$
- $\text{Alb}(M)$ = abelian variety defined by periods of 1-forms (= $\text{Jac}(C)$ for a curve)
- $i_X\omega|_A$ holomorphic 1-form on A which vanishes on $L \cap A$
- $\Rightarrow$ vanishes on B = image of $\text{Alb}(L \cap A)$.

- $H_1(B, \mathbf{Q}) \subset H_1(A, \mathbf{Q})$

- extend a symplectic basis for $H_1(B, \mathbf{Q})$ to $H_1(A, \mathbf{Q})$

- $i_X\omega$ vanishes on $B \Rightarrow$ periods vanish

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- extend a symplectic basis for $H_1(B, \mathbf{Q})$ to $H_1(A, \mathbf{Q})$

- $i_X\omega$ vanishes on $B \Rightarrow$ periods vanish

- $\Rightarrow B$ contained in zero set of flat coordinates $z_1, \ldots, z_k$

$$k = \dim B$$

- rational coefficients $\Rightarrow$ nearby varieties B in same zero set

- k constraints $\Rightarrow \dim p(L) \leq \dim \mathcal{M}/2 - k$

- $L \cap A \subset B \Rightarrow \dim(L \cap A) \leq k$

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- L Lagrangian $\Rightarrow$

  $\dim \mathcal{M}/2 = \dim L = \dim(L \cap A) + \dim p(L) \leq k + (\dim \mathcal{M}/2 - k)$

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- $\Rightarrow \dim(L \cap A) = \dim B$

  $\Rightarrow$ each component of $L \cap A$ is an abelian subvariety

- L is fibred over an affine linear subspace of $\mathcal{B}$ ...

  ... by a disjoint union of translates of an abelian subvariety

- SYZ mirror is fibred by abelian subvarieties B^0 over $p(L)$

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- SYZ mirror is fibred by abelian subvarieties B^0 over $p(L)$

- holomorphic symplectic, integrable system, ...

- but is it a hyperkähler submanifold?

ABELIAN SUBVARIETIES

EXAMPLE

- $B = \pi^* \mathrm{Jac}(\Sigma) \subset \mathrm{Jac}(S)$

- $\theta \in H^0(S, K)$ annihilates B if $a_1 = 0$

  $\det(x - \Phi) = x^n + a_1 x^{n-1} + \ldots + a_n$

- Lagrangian $L = \mathrm{Jac}(\Sigma) \times H^0(\Sigma, K^2) \oplus \cdots \oplus H^0(\Sigma, K^n)$

- $B^0 = \mathrm{P}(S, \Sigma) \Rightarrow$ mirror is $SU(n)$ moduli space

  $=$ hyperkähler submanifold

- $B \subset$ Jac(S) abelian subvariety

- $S \subset$ Jac(S), f : $S \to$ Jac(S)/B

- image curve, f factors through normalization C and Alb(C)

- $f : S \to C$

- $i_X\omega$ vanishes on B
- $\Rightarrow i_X\omega$ pulled back from A/B
- $\Rightarrow \theta = f^*\varphi$ for a 1-form φ on C

EXAMPLE $SU(2)$

- spectral curve $x^2 + a_2 = 0$, involution $x \mapsto -x$

- π^* Jac(Σ) $\subset B \subset$ Jac(S), involution on Jac(S)$/B$

  $\Rightarrow$ involution on C

  $\Rightarrow$ C is a spectral curve for a curve $\bar{\Sigma}$, $h : \Sigma \to \bar{\Sigma}$

- pull-back gives a hyperkähler submanifold.

CONCLUSIONS

- "most" $\mathbf{C}^*$-invariant Lagrangians meet a smooth fibre in dimension zero

  $\Rightarrow$ support of mirror is whole moduli space

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- "most" $\mathbf{C}^*$-invariant Lagrangians meet a smooth fibre in dimension zero

  $\Rightarrow$ support of mirror is whole moduli space

- ... but "many" hyperkähler submanifolds do not intersect the smooth fibres: if $h : \Sigma \to \bar{\Sigma}$ is ramified then S is singular