UNIVERSITY OF
OXFORD
Mathematical
Institute

Higgs bundles and mirror symmetry 1

Nigel Hitchin
Mathematical Institute
University of Oxford

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Oxford
Mathematics

HODGE THEORY

- Σ compact Riemann surface $g > 1$

- G Lie group

- $H^1(\Sigma, G)$ = equivalence classes of flat G-bundles

- ... apply Hodge theory

- trivial bundle

- connection $\nabla_A = d + A$, $A \in \Omega^1(\Sigma, \mathfrak{g})$

- curvature = $dA + \frac{1}{2}[A, A] = 0$

- gauge equivalence: $g : \Sigma \to G$,

$$\nabla_A \mapsto g^{-1} \nabla_A g$$

THE ABELIAN CASE

- $H^1(\Sigma, \mathbf{C}^*) = H^1(\Sigma, \mathbf{C})/H^1(\Sigma, \mathbf{Z}) \cong (\mathbf{C}^*)^{2g}$ (holonomy)

- Hodge theory: $H^1(\Sigma, \mathbf{C}) = H^{10} \oplus H^{01}$

- flat connection $d + \alpha^{10} + \alpha^{01}$ $\qquad \alpha^{10} = a dz$ holomorphic

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- $\beta, \gamma \in H^{10}$, flat $U(1)$-connection $\nabla_A = d + \beta - \bar{\beta}$

- $\nabla_A + \gamma + \bar{\gamma} = d + \alpha^{10} + \alpha^{01}$

  if $\beta + \gamma = \alpha^{10}, -\beta + \gamma = \overline{\alpha^{01}}$

- flat $U(1)$ connection $\nabla_A = d + \beta - \bar{\beta}$

- $\bar{\partial}_A = \bar{\partial} - \bar{\beta}$ holomorphic line bundle

- flat $\mathbf{C}^*$-connection $\cong$

  holomorphic line bundle + holomorphic 1-form

- $H^1(\Sigma, \mathbf{C}^*) \cong \text{Jac}(\Sigma) \times H^0(\Sigma, K)$

- $(\gamma, \nabla_A) \in H^0(\Sigma, K) \times \mathrm{Jac}(\Sigma) \cong H^1(\Sigma, \mathbf{C}^*)$ *not holomorphic*

- $\int_\Sigma \beta \wedge \bar\beta + \gamma \wedge \bar\gamma$ flat metric

- $\int_\Sigma \bar\beta \wedge \gamma$ holomorphic symplectic structure

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- hyperkähler metric, complex structures I, J, K

  $I^2 = J^2 = K^2 = IJK = -1$

- I: $H^0(\Sigma, K) \times \text{Jac}(\Sigma) \cong T^*\text{Jac}(\Sigma)$

  J: $H^1(\Sigma, \mathbf{C}^*) \cong (\mathbf{C}^*)^{2g}$

- $0 \to H^1(\Sigma, \mathbf{R}) \to H^1(\Sigma, \mathbf{R}^*) \to \mathbf{Z}_2^{2g} \to 0$

- $\alpha^{10} + \overline{\alpha^{10}} \in H^1(\Sigma, \mathbf{R})$

- $p : H^0(\Sigma, K) \times \mathrm{Jac}(\Sigma) \to H^0(\Sigma, K)$

- $H^1(\Sigma, \mathbf{R}^*) = 2^{2g}$ *holomorphic* sections of p

  $H^1(\Sigma, \mathbf{R}^*) =$ *real* points of $(\mathbf{C}^*)^{2g}$

HIGGS BUNDLES

- $H^1(\Sigma, GL(n,\mathbf{C})) \sim \mathrm{Hom}(\pi_1(\Sigma), GL(n,\mathbf{C}))/GL(n,\mathbf{C})$

- flat rank n vector bundle V

- irreducible reps $\Rightarrow$ smooth complex manifold

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- Hodge theory $\Rightarrow$ represent as $\nabla_A + \Phi + \Phi^*$

  ∇_A unitary connection, $\Phi \in H^0(\Sigma, \mathsf{End}\, V \otimes K)$

  (Corlette, Donaldson, Simpson, NJH,...)

- Higgs field: $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

- $\nabla_A + \Phi + \Phi^*$ flat $\Rightarrow$ $F_A + [\Phi, \Phi^*] = 0$

  Higgs bundle equations

- stability $\Rightarrow$ existence

- moduli space $\mathcal{M}$ hyperkähler

- Kähler forms $\omega_1, \omega_2, \omega_3$ for complex structures I, J, K

- holomorphic symplectic form $\omega^c = \omega_2 + i\omega_3$ relative to I etc.

- $\Rightarrow$ covariant constant holomorphic volume form

Calabi-Yau

MIRROR SYMMETRY

- symplectic geometry: A-model
- complex geometry: B-model
- + Calabi-Yau

ELSEVIER

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NUCLEAR PHYSICS B

Mirror symmetry is T-duality

Andrew Strominger[a,1], Shing-Tung Yau[b,2], Eric Zaslow[b,3]

[a] *Department of Physics, University of California, Santa Barbara, CA 93106, USA*

[b] *Department of Mathematics, Harvard University, Cambridge, MA 02138, USA*

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SYZ MIRROR SYMMETRY

- Calabi-Yau manifold M^n: ω symplectic form,

  $\Omega =$ real part of a holomorphic n-form

- special Lagrangian fibration: $p : M \to B$

  (ω, Ω vanish on fibres)

- fibres are tori T_b

- mirror = dual fibration, fibre over b = moduli space of flat $U(1)$-bundles over T_b

- X tangent vector on $B \sim$

  section of normal bundle of fibre T_b

- $i_X\omega$ well-defined and closed on T_b since fibre Lagrangian

- $[i_X\omega] \in H^1(T_b, \mathbf{R})$

- $p^*TB \cong T_F^*$

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- homology class $A \in H_1(T_b, \mathbf{R}) \Rightarrow$ closed 1-form a on B

  .... local flat coordinates $a_i = dx_i$

- $i_X \Omega$ well-defined and closed since fibre special Lagrangian

- $[i_X \Omega] \in H^{n-1}(T_b, \mathbf{R}) \cong (H^1(T_b, \mathbf{R}))^*$

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- $p^*TB \cong T_F$ and $p^*TB \cong T_F^* \Rightarrow$ metric on B

- Hessian form $\quad g_{ij} = \dfrac{\partial^2 \phi}{\partial x_i \partial x_j}$

flat coordinates

- $\widehat{M}$ dual fibration

- Gauss-Manin connection $T\widehat{M} = T_F \oplus p^*TB = H^1(\widehat{T}_b) \oplus H^1(\widehat{T}_b)$

- $T\widehat{M} = H^1(\widehat{T}_b) \oplus iH^1(\widehat{T}_b)$ complex

- metric $\Rightarrow$ Kähler metric on $\widehat{M}$

$$\omega = \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i \wedge dy_j$$

- semi-flat metric –

  Lagrangian fibres are orbits of Killing vector fields

ISSUES

- Compact Calabi-Yau threefold with a SL fibration?

- (dimension 2 – elliptic K3 – hyperkähler)

- semi-flat metric $\sim$ large complex structure limit?

- semi-flat metric not Calabi-Yau in general

HIGGS BUNDLES

- moduli space hyperkähler $\Rightarrow$ Calabi-Yau
- there is a natural special Lagrangian fibration
- the semi-flat metric is hyperkähler
- noncompactness replaces large complex structure limit

THE INTEGRABLE SYSTEM

- algebraic curve Σ

- holomorphic G^c-principal bundle P

- section $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K)$ = Higgs field

- $+$ stability condition

- $\Rightarrow$ reduction to maximal compact G
- $\Rightarrow$ G-connection A
- $F_A + [\Phi, \Phi^*] = 0$
- moduli space hyperkähler

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- holomorphic symplectic manifold

THE FIBRATION

- hyperkähler moduli space $\mathcal{M}$

  $\dim_{\mathbf{C}} = 2(g-1)\dim G^c$ (G^c semisimple)

- principal G^c-bundle, $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K)$

- invariant polynomials $p_1, \ldots, p_\ell$ on $\mathfrak{g}$

  $p_m(\Phi) \in H^0(\Sigma, K^{d_m})$

- fibration $\mathcal{M}^{2k} \to \mathbf{C}^k$

- these k functions Poisson-commute

- integrable system

- generic fibre abelian variety A

- $G^c = GL(n, \mathbf{C})$ $\det(x - \Phi) = 0$ spectral curve S

- fibre = Jac(S)

- Hyperkähler manifold + holomorphic Lagrangian fibration wrt I

- $\Rightarrow (\omega_2 + i\omega_3)$ vanishes on fibres

- ω_2 vanishes and real or imaginary part of

  $(\omega_3 + i\omega_1)^k$ vanishes

- $\Rightarrow$ special Lagrangian fibration for complex structure J

$$\omega = \omega_2,\ \Omega = \text{real or imaginary part of } (\omega_3 + i\omega_1)^k$$

$$G = GL(n, \mathbf{C})$$

- $\det(x-\Phi) = x^n + a_1 x^{n-1} + \ldots + a_n,\ a_k \in H^0(\Sigma, K^k)$

- $(a_1, \ldots, a_n) \in H^0(\Sigma, K) \oplus \cdots \oplus H^0(\Sigma, K^n) = \mathcal{B}$ base

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- spectral curve S: $\det(x - \Phi) = 0$

  $\pi : S \to \Sigma$ degree n

- x eigenvalue of $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$, $x \in H^0(S, \pi^* K)$

- x embeds S in the cotangent bundle K of Σ

- line bundle L on S

- direct image: $H^0(U, \pi_* L) \stackrel{\text{def}}{=} H^0(\pi^{-1}(U), L)$

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- $\pi : S \to \Sigma$ degree $n \Rightarrow H^0(\pi^{-1}(U), L) = H^0(U, V)$

  V rank n vector bundle

- $x \in H^0(S, \pi^* K)$
- $x : H^0(\pi^{-1}(U), L) \to H^0(\pi^{-1}(U), L \otimes \pi^* K))$
- $= \Phi : H^0(U, V) \to H^0(U, V \otimes K)$ i.e. $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

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- $x : H^0(\pi^{-1}(U), L) \to H^0(\pi^{-1}(U), L \otimes \pi^* K))$

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- fibre for a smooth curve $\sim$

  line bundles L on S: Jacobian Jac(S)

- Higgs bundle moduli space $p : \mathcal{M} \to B$
- *generic* fibre smooth abelian variety
- $B^{\mathsf{reg}} \sim$ smooth spectral curve

THE SEMI-FLAT METRIC

- M complex symplectic manifold (Kähler)

- $p : M \to B$ holomorphic Lagrangian fibration

- ... base B has a “special Kähler structure”

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- flat affine connection ∇ such that ...

  (i) Kähler form ω is covariant constant

  (ii) complex structure $I \in T \otimes T^*$ is locally ∇X

D.Freed, *Special Kähler manifolds*, Commun.Math.Phys. **203** (1999) 31–52.

- tangent vector on $B \sim$ holomorphic one-form on fibre

- cohomology complex isomorphism $H^{10}(\text{fibre}) \cong p^*TB$

- flat holomorphic coordinates $z_1, \ldots, z_N$ on B

  $\sim$ flat connection ∇

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- cohomology complex isomorphism $H^{10}(\text{fibre}) \cong p^*TB$

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  $\sim$ flat connection ∇

- Kähler $\sim$ cohomology class in $H^{11}(\text{fibre})$

  $\sim$ constant Kähler form

- semi-flat metric is hyperkähler

$$\omega_2 = \sum \frac{\partial^2 \phi}{\partial x_j \partial x_k} dx_j \wedge dy_k$$

$$\omega_1 + i\omega_3 = \frac{1}{2} \sum \omega_{jk} d(x_j + iy_j) \wedge d(x_k + iy_k)$$

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$$\omega_1 + i\omega_3 = \frac{1}{2} \sum \omega_{jk} d(x_j + iy_j) \wedge d(x_k + iy_k)$$

- defined over B^{reg} = open set of base giving smooth S

- $\mathcal{M}$ non-compact

- instead of large complex structure limit

- ... consider $\|\Phi\|^2 \to \infty$

- hyperkähler metric is approximated by the semi-flat metric

(R.Mazzeo, J.Swoboda, H.Weiss, F.Witt, *Asymptotic geometry of the Hitchin metric*, arXiv:1709.03433)

THE CIRCLE ACTION

- Higgs bundle (A, Φ)

  $\bar{\partial}_A \Phi = 0, F_A + [\Phi, \Phi^*] = 0$

- equations preserved under $(A, \Phi) \mapsto (A, e^{i\theta}\Phi)$

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  $\bar{\partial}_A \Phi = 0, F_A + [\Phi, \Phi^*] = 0$

- equations preserved under $(A, \Phi) \mapsto (A, e^{i\theta}\Phi)$

- isometry of hyperkähler metric

  $\omega_1 \mapsto \omega_1, \omega_2 + i\omega_3 \mapsto e^{i\theta}(\omega_2 + i\omega_3)$

- I-holomorphic $\mathbf{C}^*$-action $(V, \Phi) \mapsto (V, \lambda\Phi)$

- Killing vector field X infinitesimal action

- $\mathcal{L}_X\omega_1 = 0$, $\mathcal{L}_X\omega_2 = \omega_3$, $\mathcal{L}_X\omega_3 = -\omega_2$

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- $i_X(\omega_2 + i\omega_3) = i_X\omega$ holomorphic 1-form

- take a symplectic basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ for $H_1(\text{fibre}, \mathbf{Z})$

  $z_i = \int_{a_i} i_X\omega$ local flat holomorphic coordinates

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NB: $H^0(\Sigma, K^d)$ homogeneous of degree d

z_i homogeneous of degree 1

- fibre $\cong$ Jac(S)

  Abel-Jacobi map $S \to$ Jac(S)

- $H^{10}(S) \cong H^{10}(\text{Jac}(S))$, $H_1(S, \mathbf{Z}) \cong H_1(\text{Jac}(S), \mathbf{Z})$

- $S \subset$ cotangent bundle of Σ

- periods of $i_X\omega \sim$ periods of θ

  $\theta =$ canonical 1-form on cotangent bundle

- potential for special Kähler metric on space of curves:

$$K = -\frac{i}{4}\int_S \theta \wedge \bar{\theta}$$

- this defines the semi-flat metric on $\mathcal{M}$

(D.Baraglia & Z.Huang, *Special Kähler geometry of the Hitchin system and topological recursion*, arXiv:1707.04975v2)

- **Prop**:The moduli space of compact complex Lagrangian submanifolds of a holomorphic symplectic Kähler manifold has a natural special Kähler structure

  (NJH, *The moduli space of complex Lagrangian submanifolds*, Asian Journal of Mathematics, **3** (1999) 77– 92.)

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- in general $\alpha, \beta \in H^{10}(L)$, Kähler form ω

$$\int_L \alpha \wedge \bar{\beta} \wedge \omega^{n-1}$$

  defines the flat Kähler form on moduli space

  $\dim L = 1$ ω not needed