HIGGS BUNDLES AND HIGHER TEICHMÜLLER SPACES

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Abstract. In these series of lectures, we consider the moduli space of G-Higgs bundles over a compact Riemann surface X, and its correspondence with the moduli space of representations of the fundamental group of X in G, where G is a semisimple real Lie group. Some connected components of the moduli space are mundane in the sense that they are distinguished only by obvious topological invariants or have no special characteristics. Others are more alluring and unusual either because they are not detected by primary invariants, or because they have special geometric significance, or both. We will focus on such special components — generally referred as higher Teichmüller components — for Lie groups of various types including split groups (Hitchin components), hermitian groups (maximal Toledo components), etc. We will finish with a general proposal for the identification of these components in Higgs bundle terms, and its relation to the notion of positivity for Anosov representations recently developed by Guichard–Wienhard.

References

[1] M. Aparicio-Arroyo, S.B. Bradlow, B. Collier, O. García-Prada, P.B. Gothen, and A. Oliveira. Exotic components of $\mathrm{SO}(p,q)$ surface group representations, and their Higgs bundle avatars, *Comptes Rendus Mathématiques*, **356** (2018) 666–673.

[2] M. Aparicio-Arroyo, S.B. Bradlow, B. Collier, O. García-Prada, P.B. Gothen, and A. Oliveira. $\mathrm{SO}(p,q)$-Higgs bundles and higher Teichmüller components, arXiv:1802.08093v1.

[3] O. Biquard, O. García-Prada and R. Rubio, Higgs bundles, Toledo invariant and the Cayley correspondence, *Journal of Topology* **10** (2017) 795–826.

[4] S.B. Bradlow, O. García-Prada, and P.B. Gothen, Surface group representations and $\mathrm{U}(p,q)$-Higgs bundles. *J. Differential Geom.*, **64** (2003) 111–170.

[5] ———, Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces. *Geom. Dedicata*, **122** (2006) 185–213.

[6] M. Burger, A. Iozzi, and A. Wienhard. Surface group representations with maximal Toledo invariant. *Ann. of Math. (2)* **172** (2010) 517–566.

[7] K. Corlette, Flat G-bundles with canonical metrics, *J. Differential Geom.*, **28** (1988) 361–382.

[8] S.K. Donaldson, Twisted harmonic maps and the self-duality equations. *Proc. London Math. Soc.* (3) **55** (1987) 127–131.

[9] A. Domic and D. Toledo, The Gromov norm of the Kaehler class of symmetric domains. *Math. Ann.* **276** (1987) 425–432.

[10] O. García-Prada, P. B. Gothen, and I. Mundet i Riera. The Hitchin-Kobayashi correspondence, Higgs pairs and surface group representations, `arXiv:0909.4487`.

[11] ———, Higgs bundles and surface group representations in the real symplectic group, *Journal of Topology*, **6** (2013) 64–118.

[12] O. García-Prada, A. Peón-Nieto and S. Ramanan, Higgs bundles for real groups and the Hitchin–Kostant–Rallis section. *Transactions of the AMS*, **370** (2018) 2907–2953.

[13] O. Guichard and A. Wienhard, Positivity and higher Teichmüller theory, *Proceedings of the 7th European Congress of Mathematics, 2016.*

[14] W.M. Goldman, Topological components of spaces of representations. *Invent. Math.* **93** (1988) 557–607.
[15] N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc.*, **55** (1987) 59–126.
[16] ______, Stable bundles and integrable systems, *Duke Math. J.* **54** (1987) 91–114.
[17] ______, Lie groups and Teichmüller space. *Topology*, **31** (1992) 449–473.
[18] F. Labourie, Anosov flows, surface groups and curves in projective space, *Invent. Math.* **165** (2006) 51–114.
[19] J. W. Milnor, On the existence of a connection with curvature zero, *Commm. Math. Helv.* **32** (1958) 215–223.
[20] C. T. Simpson, Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization, *J. Amer. Math. Soc.* **1** (1988), 867–918.
[21] C.T. Simpson, Higgs bundles and local systems, *Publ. Math. I.H.E.S.* **75** (1992) 5–95.
[22] D. Toledo. Representations of surface groups in complex hyperbolic space. *J. Differential Geom.*, **29** (1989) 125–133.

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