Topological mirror symmetry for Hitchin systems

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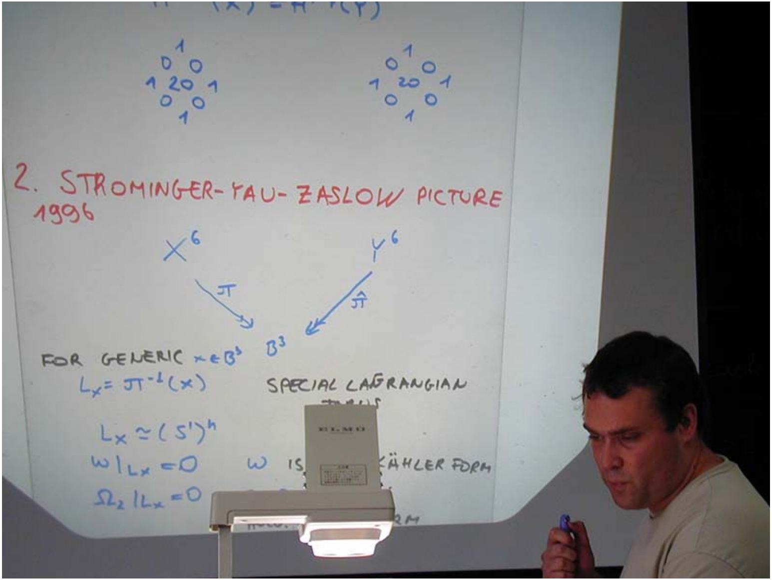
Talk on the same subject, RIMS, Kyoto 6 September 2001



Mirror Symmetry

- phenomenon first arose in various forms in string theory
- mathematical predictions (Candelas-de la Ossa-Green-Parkes 1991)
- mathematically it relates the symplectic geometry of a Calabi-Yau manifold X^d to the complex geometry of its mirror Calabi-Yau Y^d
- first aspect is the *topological mirror test* $h^{p,q}(X) = h^{d-p,q}(Y)$
- compact hyperkähler manifolds satisfy $h^{p,q}(X) = h^{d-p,q}(X)$
- (Kontsevich 1994) suggests *homological mirror symmetry* $\mathcal{D}^b(Fuk(X,\omega)) \cong \mathcal{D}^b(Coh(Y,I))$
- (Strominger-Yau-Zaslow 1996) suggests a geometrical construction how to obtain Y from X
- many predictions of mirror symmetry have been confirmed - no general understanding yet

Hodge diamonds of mirror Calabi-Yaus

Fermat quintic X

1
0 0
0 1 0
1 101 101 1
0 1 0
0 0
1

$\hat{X} := X/(\mathbb{Z}_5)^3$

1
0 0
0 101 0
1 1 1 1
0 101 0
0 0
1

K3 surface X

1
0 0
1 20 1
0 0
1

$\hat{X}$ mirror K3

1
0 0
1 20 1
0 0
1

Hitchin systems for SL_n and PGL_n Higgs bundles

- Λ degree 1 line bundle on smooth complex projective curve C
- Λ-twisted SL_n-Higgs bundle (E, ϕ)
  1. $\mathrm{rank}(E) = n \det(E) = \Lambda$
  2. $\phi \in H^0(C; \mathrm{End}_0(E) \otimes K)$
- $\check{\mathcal{M}}$ moduli space of stable SL_n-Higgs bundles, non-singular and hyperkähler
- $\Gamma = \mathrm{Pic}_C[n] \cong \mathbb{Z}_n^{2g}$ acts on $\check{\mathcal{M}}$ by tensoring $\Rightarrow$ $\hat{\mathcal{M}} := \check{\mathcal{M}}/\Gamma$ PGL_n-Higgs moduli space is an orbifold
- $$\begin{array}{rccc} \check{\mathrm{h}}: & \check{\mathcal{M}} & \to & \mathcal{A} := H^0(C, K^2) \oplus \cdots \oplus H^0(C, K^n) \\ & (E, \phi) & \mapsto & \mathrm{charpol}(\phi) = \mathrm{t}^{\mathrm{n}} + \mathrm{a}_2 \mathrm{t}^{\mathrm{n}-2} + \cdots + \mathrm{a}_{\mathrm{n}} \end{array}$$
- $\rightsquigarrow \hat{\mathrm{h}} : \hat{\mathcal{M}} \to \mathcal{A}$

Theorem (Hitchin 1987)

The Hitchin systems $\check{\mathrm{h}}, \hat{\mathrm{h}}$ are completely integrable, proper, Hamiltonian systems.

SYZ for SL_n and PGL_n Higgs bundles

- $\mathcal{C} \subset T^*C \times \mathcal{A} \to \mathcal{A}$ universal spectral curve for $\check{h} : \check{\mathcal{M}} \to \mathcal{A}$
- $\check{\mathcal{P}} := Prym_{\mathcal{A}}(\mathcal{C}) \to \mathcal{A}$ universal Prym variety
- $\check{\mathcal{P}} \circlearrowright \check{\mathcal{M}}$ over $\mathcal{A}$, when $a \in \mathcal{A}_{sm}$ $\check{\mathcal{P}}_a \circlearrowright \check{\mathcal{M}}_a$ is a $\check{\mathcal{P}}_a$-torsor
- $\Gamma = Pic(C)[n]$ acts on $\check{\mathcal{P}}$
- $\hat{\mathcal{P}} := \check{\mathcal{P}}/\Gamma \to \mathcal{A}$
- $\hat{\mathcal{P}} \circlearrowright \hat{\mathcal{M}}$ over $\mathcal{A}$, when $a \in \mathcal{A}_{sm}$ $\hat{\mathcal{P}}_a \circlearrowright \hat{\mathcal{M}}_a$ is a $\hat{\mathcal{P}}_a$-torsor
- $\check{B} \in H^2(\check{\mathcal{M}}, Z(\mathrm{SL}_n)) \subset H^2(\check{\mathcal{M}}, U(1))$ gerbe given by liftings of the universal projective bundle to an SL_n bundle
- $\hat{B} \in H^2_\Gamma(\check{\mathcal{M}}, U(1)) = H^2(\hat{\mathcal{M}}, U(1))$ Γ- equivariant liftings

Theorem (Hausel–Thaddeus 2003)

For $a \in \mathcal{A}_{sm}$

1. $\check{\mathcal{P}}_a^\vee \cong \hat{\mathcal{P}}_a$ *are dual Abelian varieties*
2. $\check{B}|_{\check{\mathcal{M}}_a}$ *and* $\hat{B}|_{\hat{\mathcal{M}}_a}$ *trivial*
3. $Triv^{U(1)}(\check{\mathcal{M}}_a, \check{B}) \cong \hat{\mathcal{M}}_a$ *and* $Triv^{U(1)}(\hat{\mathcal{M}}_a, \hat{B}) \cong \check{\mathcal{M}}_a$

Topological mirror test

- $$\check{\mathcal{M}} \xrightarrow{\check{h}} \mathcal{A} \xleftarrow{\hat{h}} \hat{\mathcal{M}}$$
- [Hausel–Thaddeus 2003] $\rightsquigarrow$ generic fibers torsors for dual abelian varieties $\rightsquigarrow$ hyperkähler rotation satisfies SYZ
- In the first two lectures we will discuss the mirror symmetry proposal of [Hausel–Thaddeus 2003]:
  ”Hitchin systems for Langlands dual groups satisfy Strominger-Yau-Zaslow, so could be considered mirror symmetric; in particular they should satisfy the *topological mirror tests*:”

Conjecture (Hausel–Thaddeus 2003, ”Topological mirror test”)

$$\begin{aligned} h^{p,q}(\check{\mathcal{M}}) &= h^{p,q}_{\mathrm{st}}(\hat{\mathcal{M}}, \hat{B}) \\ &= h^{p,q}_{\mathrm{st}}(\check{\mathcal{M}}/\Gamma, \hat{B}) := \sum_{\gamma \in \Gamma} h^{p-F(\gamma),q-F(\gamma)}(\check{\mathcal{M}}^{\gamma}/\Gamma, L_{\hat{B}}) \end{aligned}$$

E-polynomials

- (Deligne 1972) constructs weight filtration $W_0 \subset \cdots \subset W_k \subset \cdots \subset W_{2d} = H_c^d(X;\mathbb{Q})$ for any complex algebraic variety X, plus a pure Hodge structure on W_k/W_{k-1} of weight k
- we say that the weight filtration is *pure* when $W_k/W_{k-1}(H_c^i(X)) \neq 0 \Rightarrow k = i$; examples include smooth projective varieties, $\check{\mathcal{M}}$ and $\hat{\mathcal{M}}$
- define $E(X;x,y) := \sum_{i,j,d} (-1)^d x^i y^j h^{i,j}\left(W_k/W_{k-1}(H_c^d(X,\mathbb{C}))\right)$
- basic properties:
  additive - if $X_i \subset X$ locally closed s.t. $\dot\cup X_i = X$ then $E(X;x,y) = \sum E(X_i;x,y)$
  multiplicative - $F \to E \to B$ locally trivial in the Zariski topology $E(E;x,y) = E(B;x,y)E(F;x,y)$
- when weight filtration is pure then $h^{p,q}(X) := h^{p,q}(H_c^{p+q}(X))$ and $E(X;-x,-y) = \sum_{p,q} h^{p,q}(X)x^p y^q$ is the Hodge $E(X;t,t)$ is the Poincaré polynomial

Stringy E-polynomials

- let finite group Γ act on a non-singular complex variety M
- $E_{st}(M/\Gamma; x, y) := \sum_{[\gamma]\in[\Gamma]} E(M_\gamma/C(\gamma); x, y)(xy)^{F(\gamma)}$ *stringy E-polynomial*
- $F(\gamma)$ is the fermionic shift, defined as $F(\gamma) = \sum w_i$, where γ acts on $TX|_{X_\gamma}$ with eigenvalues $e^{2\pi i w_i}$, $w_i \in [0, 1)$
- $F(\gamma)$ is an integer when M is CY and Γ acts trivially on K_M
- motivating property [Kontsevich 1995] if $f : X \to M/\Gamma$ crepant resolution $\Leftrightarrow K_X = f^* K_{M/\Gamma}$ then $E(X; x, y) = E_{st}(M/\Gamma; x, y)$
- if $B \in H^2_\Gamma(M, U(1))$ is a Γ-equivariant flat $\mathrm{U}(1)$-gerbe on M, then on each $\mathcal{M}_\gamma$ we get an automorphism of $B|_{\mathcal{M}_\gamma} \rightsquigarrow$ $C(\gamma)$-equivariant local system $L_{B,\gamma}$
- we can define $E^B_{st}(M/\Gamma; x, y) := \sum_{[\gamma]\in[\Gamma]} E(M_\gamma, L_{B,\gamma}; x, y)^{C(\gamma)}(xy)^{F(\gamma)}$ *stringy E-polynomial twisted by a gerbe*

Topological mirror symmetry for the toy model

- let $\check{\mathcal{M}}$ be moduli space of SL_2 parabolic Higgs bundles on elliptic curve E with one parabolic point
- $\mathbb{Z}_2$ acts on E and $\mathbb{C}$ as additive inverse $x \mapsto -x$
- $\check{\mathcal{M}} \to (E \times \mathbb{C})/\mathbb{Z}_2$ blowing up; $\mathrm{h} : \check{\mathcal{M}} \to \mathbb{C}/\mathbb{Z}_2 \cong \mathbb{C}$ is elliptic fibration with $\hat{D}_4$ singular fiber over 0
- $\Gamma = E[2] \cong \mathbb{Z}_2^2$ acts on $\check{\mathcal{M}}$ by multiplying on E
- $\hat{\mathcal{M}}$ the PGL_2 moduli space is $\check{\mathcal{M}}/\Gamma$ an orbifold, elliptic fibration over $\mathbb{C}$ with A_1 singular fiber
  with three $\mathbb{C}^2/\mathbb{Z}_2$-orbifold points on one of the components
- blowing up the three orbifold singularities is crepant gives $\check{\mathcal{M}}$
- topological mirror test: $E_{st}(\hat{\mathcal{M}}; x, y) \overset{Kontsevich}{=} E(\check{\mathcal{M}}; x, y)$

Toy model

Topological mirror symmetry conjecture - unravelled

Conjecture (Hausel–Thaddeus, 2003)

$E(\check{\mathcal{M}}) = E_{st}^{\hat{B}}(\hat{\mathcal{M}})$

- Proved for $n = 2, 3$ using [Hitchin 1987] and [Gothen 1994].
- as Γ acts on $H^*(\check{\mathcal{M}})$ we have $\rightsquigarrow$
  $H^*(\check{\mathcal{M}}) \cong \oplus_{\kappa \in \hat{\Gamma}} H^*_\kappa(\check{\mathcal{M}}) \rightsquigarrow$

$$\begin{array}{ccccccc} E(\check{\mathcal{M}}) & = & \sum_{\kappa\in\hat{\Gamma}} E_\kappa(\check{\mathcal{M}}) & = & E_0(\check{\mathcal{M}}) & + & \overbrace{\sum_{\kappa\in\hat{\Gamma}^*} E_\kappa(\check{\mathcal{M}})}^{\textit{variant}} \\ & & & & \| & & \\ E^B_{st}(\hat{\mathcal{M}}) & = & \sum_{\gamma\in\Gamma} E(\check{\mathcal{M}}_\gamma, L_{B,\gamma})^\Gamma & = & E(\check{\mathcal{M}})^\Gamma & + & \underbrace{\sum_{\gamma\in\Gamma^*} E(\check{\mathcal{M}}_\gamma/\Gamma, L_{B,\gamma})}_{\textit{stringy}} \end{array}$$

- $\Gamma \cong H^1(C, \mathbb{Z}_n)$ and wedge product induces $w : \Gamma \cong \hat{\Gamma}$
- refined Topological Mirror Test for $w(\gamma) = \kappa$:
  $E_\kappa(\check{\mathcal{M}}) = E(\check{\mathcal{M}}_\gamma/\Gamma, L_{B,\gamma})$

Example SL_2

- fix $n = 2$
- $\mathbb{T} := \mathbb{C}^\times$ acts on $\check{\mathcal{M}}$ by $\lambda \cdot (E, \phi) \mapsto (E, \lambda \cdot \phi) \overset{\textit{Morse}}{\rightsquigarrow}$

$$H^*(\check{\mathcal{M}}) = \bigoplus_{F_i \subset \check{\mathcal{M}}^{\mathbb{T}}} H^{*+\mu_i}(F_i) \qquad \text{as } \Gamma\text{-modules}$$

- $F_0 = \check{\mathcal{N}}$ where $\phi = 0$; then [Harder–Narasimhan 1975] $\Rightarrow$ $H^*(F_0)$ is trivial Γ-module
- for $i = 1, \ldots, g-1$

$$F_i = \{(E, \phi) \mid E \cong L_1 \oplus L_2, \phi = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}, \varphi \in H^0(L_1^{-1} L_2 K)\}$$

$\rightsquigarrow F_i \to S^{2g-2i-1}(C)$ Galois cover with Galois group Γ

Theorem (Hitchin 1987)

The Γ action on $H^*(F_i)$ is only non-trivial in the middle degree $2g-2i-1$. For $\kappa \in \hat{\Gamma}^*$ we have

$$\dim H_\kappa^{2g-2i-1}(F_i) = \binom{2g-2}{2g-2i-1}.$$

Example PGL_2

- $\gamma \in \Gamma = \mathrm{Pic}^0(C)[2] \rightsquigarrow C_\gamma \overset{2:1}{\to} C$ with Galois group $\mathbb{Z}_2$
- $$\begin{array}{ccccc} \mathcal{M}^1(\mathrm{GL}_1, C_\gamma) & \overset{\text{push-forward}}{\longrightarrow} & \mathcal{M}^1(\mathrm{GL}_2, C) & \supset & \check{\mathcal{M}}^1 \\ \| & & \downarrow \det & & \\ T^*\,\mathrm{Jac}^1(C_\gamma) & \overset{N_m(C_\gamma/C)}{\longrightarrow} & T^*\,\mathrm{Jac}^1(C) & \ni & (\Lambda, 0) \end{array}$$
- let $\check{\mathcal{M}}(\mathrm{GL}_1, C_\gamma) := N_m(C_\gamma/C)^{-1}(\Lambda, 0)$ *endoscopic* H_γ-Higgs moduli space
- after [Narasimhan–Ramanan, 1975]
  $\check{\mathcal{M}}_\gamma = \check{\mathcal{M}}(\mathrm{GL}_1, C_\gamma)/\mathbb{Z}_2 \cong T^*\,\mathrm{Prym}^1(C_\gamma/C)/\mathbb{Z}_2$
- can calculate $\dim H^{2g-2i+1}(\check{\mathcal{M}}_\gamma/\Gamma, L_{\hat{B},\gamma}) = \binom{2g-2}{2g-2i-1}$
  and 0 otherwise

Theorem (Hausel–Thaddeus, 2003)

when $n = 2$ and $\kappa = w(\gamma)$
$E_\kappa(\check{\mathcal{M}}) = E(\check{\mathcal{M}}_\gamma/\Gamma; L_{B,\gamma})$

p-adic integration

- $p \in \mathbb{Z}$ prime; $\mathbb{Q}_p := \{\sum_{n \geq n_0} a_n p^n : a_n \in \{0, \ldots, p-1\}\}$ the field of p-adic numbers
- $\mathbb{Q}_p$ completion of $\mathbb{Q}$ w.r.t. the p-adic absolute value $|\frac{a}{b}|_p = p^{v_p(b) - v_p(a)}$
- non-archimedian: $|a|_p \leq 1$ for every integer $a \in \mathbb{Z} \subset \mathbb{Q}$
- $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ *p-adic integers* compact subring local, with residue field $\mathbb{F}_p := \mathbb{Z}_p / p\mathbb{Z}_p$
- $\mathbb{Q}_p$ locally compact $\rightsquigarrow$ Haar measure μ, normalized $\mu(\mathbb{Z}_p) = 1$ $\rightsquigarrow$ integration $\int_{\mathbb{Q}_p} f d\mu \in \mathbb{C}$ for $f \in C_c(\mathbb{Q}_p, \mathbb{C})$
- generalizes to integration using degree n differential forms on $\mathbb{Q}_p$-analytic manifolds
- compact example: $\mathbb{Z}_p$-points of the smooth locus of a variety defined over $\mathbb{Z}_p$
- $d\mu_\omega$ measure given by the $\mathbb{Q}_p$ analytic n-form ω on an n-dimensional $\mathbb{Q}_p$ manifold
- crucial fact: $X \subset Y$ such that $\mathrm{codim}(X) \geq 1$, and ω top dimensional form on Y then $\int_{X(\mathbb{Z}_p)} d\mu_\omega = 0$

p-adic integration and Hodge numbers

Theorem (Weil 1961)

Let X be a scheme over $\mathbb{Z}_p$ of dimension n and $\omega \in H^0(X, \Omega_X^n)$ nowhere vanishing (called gauge form*). Then*

$$\int_{X(\mathbb{Z}_p)} d\mu_\omega = \frac{|X(\mathbb{F}_p)|}{p^n}.$$

- example: $X = \{(x, y) \in \mathbb{Z}_p^2 : xy = 1\}$ then $X(\mathbb{Z}_p) = \mathbb{Z}_p^\times = \{\sum_{n\geq 0} a_n p_n : a_0 \neq 0\} = \mathbb{Z}_p \setminus p\mathbb{Z}_p$, thus $\mu(X(\mathbb{Z}_p)) = \mu(\mathbb{Z}_p) - \mu(p\mathbb{Z}_p) = 1 - \frac{1}{p} = \frac{p-1}{p} = \frac{|\mathbb{F}_p^\times|}{p} = \frac{|X(\mathbb{F}_p)|}{p}$ but this is $\neq \mu(\mathbb{Z}_p \setminus \{0\}) = 1$
- [Deligne 1974] $\Rightarrow$ if X and Y defined over $R \subset \mathbb{C}$ finitely generated over $\mathbb{Z}$, and if for any $\phi : R \to \mathbb{F}_q$ we have $|X_\phi(\mathbb{F}_q)| = |Y_\phi(\mathbb{F}_q)|$ then $E(X; x, y) = E(Y; x, y)$.
- this implies a theorem of Batyrev and Kontsevich: birational smooth projective CY varieties have the same Hodge numbers

p-adic integration and stringy Hodge numbers

Theorem (Gröchenig–Wyss–Ziegler 2017)

$X = Y/\Gamma$ a CY orbifold, with gauge form ω and $B \in H^2_\Gamma(Y, \mathbb{Q}/\mathbb{Z})$ then there exists a function $f_B : X(\mathbb{Z}_p) \to \mathbb{C}$ such that

$$\int_{X(\mathbb{Z}_p)} f_B \, d\mu_\omega = \frac{\#^B_{st} X(\mathbb{F}_p)}{p^{\dim X}}$$

$$= p^{-\dim X} \sum_{[\gamma]\in[\Gamma]} p^{F(\gamma)} \sum_d Tr(Fr_p, H^d_{\acute{e}t}(Y_\gamma, L_B)^{C(\gamma)})(-1)^d$$

$\rightsquigarrow$ for CY orbifolds $X_i = Y_i/\Gamma_i$ with gauge form ω_i and gerbe B_i

$$\int_{X_1(\mathbb{Z}_p)} f_{B_1} d\mu_{\omega_1} = \int_{X_2(\mathbb{Z}_p)} f_{B_2} d\mu_{\omega_2} \forall p \Rightarrow E^{B_1}_{st}(X_1(\mathbb{C})) = E^{B_2}_{st}(X_2(\mathbb{C}))$$

- when $B_i = 0 \rightsquigarrow$ [Batyrev 1997] [Kontsevich 1995]: $X \to Y/\Gamma$ crepant resolution of CY orbifold:

$$E_{st}(Y/\Gamma) = E(X)$$

Proof of TMS via p-adic integration

- strategy of proof [Gröchenig-Wyss-Ziegler 2017]:
- recall $\check{\mathcal{M}}$ SL_n Higgs moduli space, smooth, $\check{B} \in H^2(\check{\mathcal{M}}, \mathbb{Q}/\mathbb{Z})$
- $\hat{\mathcal{M}} := \check{\mathcal{M}}/\Gamma$ PGL_n moduli space, orbifold, $\hat{B} \in H^2(\check{\mathcal{M}}, \mathbb{Q}/\mathbb{Z})$
- consider them over $\mathbb{Z}_p$
- for $a \in \mathcal{A}(\mathbb{Z}_p) \cap \mathcal{A}_{sm}(\mathbb{Q}_p)$ when $\check{\mathcal{M}}_a(\mathbb{Z}_p) \neq \emptyset$ and $\hat{\mathcal{M}}_a(\mathbb{Z}_p) \neq \emptyset$ then $f_{\check{B}}$ and $f_{\hat{B}}$ are 1 on them and $\int_{\check{\mathcal{M}}_a(\mathbb{Z}_p)} f_{\check{B}} = vol(\check{\mathcal{M}}_a(\mathbb{Z}_p)) = vol(\hat{\mathcal{M}}_a(\mathbb{Z}_p)) = \int_{\hat{\mathcal{M}}_a(\mathbb{Z}_p)} f_{\hat{B}}$
- when $\check{\mathcal{M}}_a(\mathbb{Z}_p) = \emptyset$ and $\hat{\mathcal{M}}_a(\mathbb{Z}_p) \neq \emptyset$ then Tate duality $\rightsquigarrow$ $f_{\check{B}}$ is a non-trivial character on Abelian variety $\hat{\mathcal{M}}_a(\mathbb{Z}_p) \Rightarrow$ $\int_{\hat{\mathcal{M}}_a(\mathbb{Z}_p)} f_{\hat{B}} = 0$
- $\int_{\check{\mathcal{M}}(\mathbb{Z}_p)} f_{\check{B}} d\mu_{\check{\omega}} = \int_{\check{\mathrm{h}}^{-1}(\mathcal{A}(\mathbb{Z}_p)\cap\mathcal{A}_{sm}(\mathbb{Q}_p))} f_{\check{B}} d\mu_{\check{\omega}} = \int_{\hat{\mathrm{h}}^{-1}(\mathcal{A}(\mathbb{Z}_p)\cap\mathcal{A}_{sm}(\mathbb{Q}_p))} f_{\hat{B}} d\mu_{\hat{\omega}} = \int_{\hat{\mathcal{M}}(\mathbb{Z}_p)} f_{\hat{B}} d\mu_{\hat{\omega}}$
- $\rightsquigarrow E_{st}(\check{\mathcal{M}}(\mathbb{C})) = E_{st}^{\hat{B}}(\hat{\mathcal{M}}(\mathbb{C}))$ topological mirror symmetry!
- $\rightsquigarrow E_\kappa(\check{\mathcal{M}}) = E(\check{\mathcal{M}}_\gamma/\Gamma; L_{\hat{B}})$ refined TMS
- holds along singular fibers of Hitchin map: new proof of Ngô's geometric stabilisation theorem

Comments on proof

- proof was accomplished by comparing p-adic volumes of smooth fibers of $\check{\mathrm{h}}$ and $\hat{\mathrm{h}}$ over $\mathbb{Q}_p$
- such fibers could behave badly over $\mathbb{Z}_p$
- example: $\pi : X = \mathbb{Z}_p^2 :\to \mathbb{Z}_p$ s.t. $\pi(x, y) = xy$
- all fibers of $X(\mathbb{Z}_p) \to \mathbb{Z}_p$ over $z \neq 0$ are smooth $\mathbb{Q}_p$ manifolds
- e.g. solutions of $xy = p \Rightarrow$ either $x \in p\mathbb{Z}_p^\times$ and $y \in \mathbb{Z}_p^\times$ or $x \in \mathbb{Z}_p^\times$ and $y \in p\mathbb{Z}_p^\times$: is two copies of $\mathbb{Z}_p^\times$, smooth over $\mathbb{Q}_p$
- but singular over $\mathbb{Z}_p \to \mathbb{F}_p$
- another comment: the same proof should show that $\int_{\check{\mathcal{M}}^0(\mathbb{Z}_p)} d\mu_{\check{\omega}} = \int_{\hat{\mathcal{M}}(\mathbb{Z}_p)} d\mu_{\hat{\omega}}$ where $\check{\mathcal{M}}^0$ is, singular, moduli of semistable SL_n Higgs bundles of trivial determinant
- can we define $E_{st}(\check{\mathcal{M}}^0)$ such that it will agree with $E_{st}(\hat{\mathcal{M}})$?

Mirror symmetry for Higgs bundles

- G complex reductive group G^L its Langlands dual e.g. $\mathrm{GL}_n^L = \mathrm{GL}_n$, $\mathrm{SL}_n^L = \mathrm{PGL}_n$
- C smooth complex projective curve
- $\mathcal{M}_{\mathrm{Dol}}(\mathrm{G})$ (or $\mathcal{M}(\mathrm{G})$) moduli of semi-stable G-Higgs bundles; $h_{\mathrm{G}} : \mathcal{M}_{\mathrm{Dol}}(\mathrm{G}) \to \mathcal{A}_{\mathrm{G}}$ Hitchin map
- $\mathcal{M}_{\mathrm{DR}}(\mathrm{G})$ moduli space flat G connections
- $\mathcal{M}_{\mathrm{Dol}}(\mathrm{G}) \cong_{diff} \mathcal{M}_{\mathrm{DR}}(\mathrm{G})$ non-abelian Hodge theorem (Hitchin-Donaldson-Simpson-Corlette)
- $$\mathcal{M}_{\mathrm{DR}}(\mathrm{G}) \xrightarrow{\mathrm{h}_{\mathrm{G}}} \mathcal{A}_G \cong \mathcal{A}_{G^L} \xleftarrow{\mathrm{h}_{\mathrm{G}^L}} \mathcal{M}_{\mathrm{DR}}(\mathrm{G}^L)$$

  special Lagrangian fibration, with dual generic fibres (Strominger-Yau-Zaslow 1995) mirror symmetry
- (Kapustin-Witten 2005) $S : Fuk(\mathcal{M}_{\mathrm{DR}}(\mathrm{G})) \cong D_{coh}(\mathcal{M}_{\mathrm{DR}}(\mathrm{G}^L))$ homological mirror symmetry of (Kontsevich 1994)

Semi-classical limit

- (Donagi–Pantev 2012) semi-classical limit
  $S_{sc} : D_{coh}(\mathcal{M}(\mathrm{G})) \cong D_{coh}(\mathcal{M}(\mathrm{G}^L))$
- holds along generic fibers of the Hitchin map by fiberwise Fourier-Mukai transform

$$\begin{array}{ccc} \mathcal{M}(\mathrm{G}) & & \mathcal{M}(\mathrm{G}^L) \\ & {}_{\mathrm{h_G}}\searrow \quad \swarrow_{\mathrm{h}_{\mathrm{G}^L}} & \\ & \mathcal{A}_G \cong \mathcal{A}_{G^L} & \end{array}$$

- recall $FM : D_{coh}(\mathrm{h}_{\mathrm{G}}^{-1}(a)) \cong D_{coh}(\mathrm{h}_{\mathrm{G}}^{-1}(a)^{\vee}) \cong D_{coh}(\mathrm{h}_{\mathrm{G}^L}^{-1}(a))$
- e.g. *FM* swaps skycraper sheaves with line bundles
- cohomological shadow should be (Hausel-Thaddeus 2003) for
  $\mathrm{G} = \mathrm{SL}_n \rightsquigarrow E_{st}(\mathcal{M}(\mathrm{SL}_n)) = E_{st}(\mathcal{M}(\mathrm{PGL}_n))$
  topological mirror symmetry conjecture
- proved $\mathrm{SL}_2, \mathrm{SL}_3$
- recently proved SL_n (Gröchenig–Wyss–Ziegler 2017) using p-adic integration

BAA branes

- from now on $\mathrm{G} = \mathrm{GL}_2$
- $\mathrm{U}(1,1) \subset \mathrm{GL}_2$ real form; $\mathrm{U}(1,1)$-Higgs bundle rank 2 $E \xrightarrow{\Phi} E \otimes K_C$ such that $\Phi \neq 0$ and $E \xrightarrow{\Phi} E \otimes K_C \cong E \xrightarrow{-\Phi} E \otimes K_C$
- $\mathcal{M}(\mathrm{U}(1,1)) \subset \mathcal{M}(\mathrm{GL}_2)$ has g components $L_0, \ldots, L_{g-1}$ (Schaposnik 2015, Hitchin 2016)
- L_i locus of Higgs bundles $M_1 \oplus M_2 \overset{\begin{pmatrix} 0 & \phi_1 \\ \phi_2 & 0 \end{pmatrix}}{\longrightarrow} (M_1 \oplus M_2) \otimes K_C$ with $\deg(M_1) = i - g + 1$
- for $i = 0, \ldots, g-2$ $L_i \subset \mathcal{M}(\mathrm{GL}_2)^s$ smooth, total space of vector bundle E_i on $F_i = Jac(C) \times C_{2i} = (M_1, H^0(M_2^{-1} M_1 K_C))$
- $L_i \subset \mathcal{M}(\mathrm{GL}_2)$ holomorphic Lagrangian subvarieties $\rightsquigarrow$ BAA branes
- $\mathcal{L} = \det(R\mathrm{pr}_{1*}\mathbb{E})$ determinant line bundle on $\mathcal{M}(\mathrm{GL}_2)$ from universal bundle $\mathbb{E} \xrightarrow{\Phi} \mathbb{E} \otimes K_C$ on $\mathcal{M}(\mathrm{GL}_2) \times C$
- $\mathcal{L}_i := \mathcal{L}^2 \otimes \mathcal{O}_{L_i} \in D_{coh}(\mathcal{M}(\mathrm{GL}_2))$; note $\mathcal{L}_i^2 = K_{L_i} \rightsquigarrow \mathcal{L}_i^\vee \cong \mathcal{L}_i$

BBB branes

- what is $S_{sc}(\mathcal{L}_i)$?
- first clue: S_{sc} generically Fourier-Mukai transform
- (Schaposnik 2015) $L_i \cap h^{-1}(a)$ is $\binom{4g-4}{2i}$ copies of $Jac(C)$
- $\rightsquigarrow S_{sc}(\mathcal{L}_i) \cap h^{-1}(a)$ rank $\binom{4g-4}{2i}$ vector bundle supported on $\mathcal{M}(\mathrm{SL}_2)$
- second clue: physics $\rightsquigarrow$ mirror of BAA brane should be BBB brane i.e. triholomorphic sheaf
- (Hitchin 2016) $\rightsquigarrow S_{sc}(\mathcal{L}_i) = \Lambda^{2i}(\mathbb{V}) := \Lambda^i$ should be exterior power of the Dirac-Yang-Mills bundle $\mathbb{V}$ on $\mathcal{M}(\mathrm{SL}_2)$
- $\mathbb{V} := R\mathrm{pr}_{1*}(\mathbb{E} \xrightarrow{\mathbf{\Phi}} \mathbb{E} \otimes K_C)[1]$ triholomorphic rank $4g-4$ vector bundle on $\mathcal{M}(\mathrm{SL}_2)^s$
- note Λ^i has an orthogonal structure $\rightsquigarrow \Lambda^{i\vee} \cong \Lambda^i$
- can we find computational evidence for $S_{sc}(\mathcal{L}_i) = \Lambda^i$ outside generic locus of h?
- Yes! compute $|Hom_{D_{coh}(\mathcal{M}(\mathrm{GL}_2))}(\mathcal{L}_i, \Lambda^j)| \in \mathbb{C}[[t]]$ using $\mathbb{T} = \mathbb{C}^\times$ action $(E, \Phi) \to (E, \lambda\Phi)$

Main computation

- X semi-projective $\mathbb{T}$-variety and $\mathcal{F} \in Coh_{\mathbb{T}}(X)$ denote $\chi_{\mathbb{T}}(\mathcal{F}) = \sum \dim(H^k(X;\mathcal{F})^l)(-1)^k t^{-l} \in \mathbb{C}((t))$
- $\mathbb{T}$ action $(E,\Phi) \to (E,\lambda\Phi) \rightsquigarrow \mathcal{M}(\mathrm{GL}_2)$ semi-projective
- fixed point components $F_i = Jac(C) \times C_{2i}$ for $i = 0,\dots,g-2$ and $\mathcal{N}(\mathrm{GL}_2)$ where $\Phi = 0$
- $L_i \subset \mathcal{M}(\mathrm{GL}_2)$ is the total space E_i of vector bundle on F_i
- $\Lambda_{s^2} := \bigoplus_{i=0}^{2g-2} s^{2i}\Lambda^{2i}\mathbb{V}$
- Grothendieck-Riemann-Roch $\chi_{\mathbb{T}}(L_i;\mathcal{L}_i \otimes \Lambda_{s^2}) = \chi_{\mathbb{T}}(F_i;\mathcal{L}_i \otimes \Lambda_{s^2} \otimes \mathrm{Sym}_{t^2}(E_i^*))$ $= \int_{F_i} \check{\mathrm{h}}(\mathcal{L}_i)\check{\mathrm{h}}(\Lambda_{s^2})\check{\mathrm{h}}(\mathrm{Sym}_{t^2}(E_i^*))\mathrm{td}(T_{F_i})$

Theorem (Hausel–Mellit–Pei 2017)

for $i \leq g-2$ $\qquad \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2);\mathcal{L}_i \otimes \Lambda_{s^2}) =$

$$\frac{1}{2\pi i}\oint_{|z^2-t|=\epsilon} z^{4(g-1-i)} \frac{\left((1+\frac{s}{zt})(1+sz)\right)^{2g-2-i}\left((1+\frac{s}{z})(1+\frac{sz}{t})\right)^{i}}{(1-tz^2)^{3g-3-i}(1-t/z^2)^{2i-g+1}}$$
$$\left(4+\frac{\frac{s}{z}}{1+\frac{s}{z}}+\frac{sz}{1+sz}+\frac{4tz^2}{1-tz^2}+\frac{\frac{4t}{z^2}}{1-\frac{t}{z^2}}-\frac{\frac{s}{zt}}{1+\frac{s}{zt}}-\frac{\frac{sz}{t}}{1+\frac{sz}{t}}\right)^g \frac{dz}{z} \in \mathbb{C}[[t,s]]$$

Symmetry

- from mirror symmetry we expect
  $\chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda^j) = \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_j \otimes \Lambda^i)$
- $f(z,s) := \frac{z^4(1-t/z^2)^2(1+\frac{s}{zt})(1+sz)}{(1-tz^2)^2(1+\frac{s}{z})(1+\frac{sz}{t})}$
  $h(z,s) := z\frac{\partial f}{\partial z}\frac{(1+\frac{s}{zt})(1+sz)(1+\frac{s}{z})(1+\frac{sz}{t})}{(1-\frac{t}{z^2})(-tz^2+1)}.$
- Theorem $\rightsquigarrow \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda^j) = \oint_{|s|=\epsilon}\oint_{|f|=\epsilon} \frac{h^{g-1}}{f^i s^{2j}}\frac{df}{f}\frac{ds}{s}$
- $w = \left(\frac{(sz+t)(tz+s)}{(sz+1)(s+z)}\right)^{\frac{1}{2}}$
- $f(w, f^{\frac{1}{2}}) = s^2$; $s(w, f^{\frac{1}{2}}) = f^{\frac{1}{2}}$ and $h(w, f^{\frac{1}{2}}) = h(z,s)$

Theorem (Hausel–Mellit–Pei 2017)

$\chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda^j) = \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_j \otimes \Lambda^i)$

Reflection of mirror symmetry

- semi-classical limit $S_{sc} : D_{coh}(\mathcal{M}(\mathrm{GL}_2)) \cong D_{coh}(\mathcal{M}(\mathrm{GL}_2))$
- $S_{sc}(\mathcal{L}_i) = \Lambda_i$ should imply $Hom_{D_{coh}(\mathcal{M}(\mathrm{GL}_2))}(\mathcal{L}_i, \Lambda^j) \cong Hom_{D_{coh}(\mathcal{M}(\mathrm{GL}_2))}(\Lambda^i, \mathcal{L}_j)$
- more generally define *equivariant Euler form*

$$\chi_{\mathbb{T}}(\mathcal{L}_i, \Lambda^j) := \sum_{k,l} \dim(H^k(\mathbf{R}Hom(\mathcal{L}_i, \Lambda^j))^l)(-1)^k t^{-l} \in \mathbb{C}[[t]]$$

- $\chi_{\mathbb{T}}(\Lambda^j, \mathcal{L}_i) = \chi_{\mathbb{T}}(\Lambda^{j\vee} \otimes \mathcal{L}_i) = \chi_{\mathbb{T}}(\Lambda^j \otimes \mathcal{L}_i)$
- $\chi_{\mathbb{T}}(\mathcal{L}_j, \Lambda^i) = \chi_{\mathbb{T}}(\mathcal{L}_j^\vee \otimes \Lambda^j) = \chi_{\mathbb{T}}(\mathcal{L}^{-2} \otimes \mathcal{O}_{L_j} \otimes K_{L_j} \otimes \Lambda^i) = \chi_{\mathbb{T}}(\mathcal{L}_j \otimes \Lambda^i)$

Theorem (Hausel–Mellit–Pei)

We have $\chi_{\mathbb{T}}(\Lambda^j, \mathcal{L}_i) = \chi_{\mathbb{T}}(\mathcal{L}_j, \Lambda^i)$

Problems

- Maple indicates $\chi_{\mathbb{T}}(\mathcal{L}_i, \mathcal{L}_j) = \chi_{\mathbb{T}}(\Lambda^i, \Lambda^j)$, can we prove it?
- another conjectured mirror (Neitzke) is $S_{sc}(\mathcal{L}) = \mathcal{L}^{-1}$, this should imply $\chi_{\mathbb{T}}(\mathcal{L}^{\pm 1} \otimes \mathcal{L}_j) = \chi_{\mathbb{T}}(\mathcal{L}^{\mp 1} \otimes \Lambda^j)$. Maple agrees
- (Hitchin 2016) proposes similar story for $\mathrm{U}(m, m) \subset \mathrm{GL}_{2m}$ with mirror being the Dirac bundle supported on $\mathcal{M}(Sp(m))$. Can our computations generalize to that case?
- Can we find computational evidence that $S_{sc}(\mathcal{L}_H)$, i.e. H-Higgs bundles for a real form $H \subset G$ has support in $\mathcal{M}_{\mathrm{Dol}}(N_H) \subset \mathcal{M}_{\mathrm{Dol}}(\mathrm{G}^L)$ where $N_H \subset G^L$ Nadler group of H as in (Baraglia, Schaposnik 2016)?
- Do these observations fit into the TQFT framework for $\chi_{\mathbb{T}}(\mathcal{M}(G), \mathcal{L}^k)$? What is $S_{sc}(\mathcal{L}^k)$?
- How do Wilson and t'Hooft operators act on $\mathcal{L}_i, \Lambda^j$?
- Could we guess FM transform on $K_{\mathbb{T}}(h^{-1}(0))$ for the nilpotent cone?