Higgs bundles and higher Teichmüller spaces

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1. Moduli space of representations

- S oriented smooth compact surface of genus $g \geq 2$
- $\pi_1(S)$ fundamental group of S
- G connected semisimple Lie group (real or complex)

A representation of $\pi_1(S)$ in G

is a homomorphism

$$\rho\colon \pi_1(S) \to G$$

- $\mathrm{Hom}(\pi_1(S), G)$ is an analytic variety, which is algebraic if G is algebraic
- G acts on $\mathrm{Hom}(\pi_1(S), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1} \text{ for } g \in G, \rho \in \mathrm{Hom}(\pi_1(S), G)$$

- ρ is a **reductive representation** if composed with the adjoint representation in the Lie algebra of G, decomposes as a sum of irreducible representations
- $\mathrm{Hom}^+(\pi_1(S), G)$: set of reductive representations

Moduli space of representations or character variety

The moduli space of representations of $\pi_1(S)$ in G is defined to be the orbit space

$$\mathcal{R}(S, G) = \mathrm{Hom}^+(\pi_1(S), G)/G$$

- $\mathcal{R}(S, G)$ is an analytic variety (algebraic if G is algebraic)
- Interested in **the topology and geometry** of $\mathcal{R}(S, G)$
- Complex algebraic geometry approach: **Higgs bundles**

2. Higgs bundles

- X compact Riemann surface
- $H \subset G$ maximal compact subgroup of G
- θ Cartan involution of $\mathfrak{g}$, Lie algebra of G, defining the **Cartan decomposition**:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$$

  where $\mathfrak{h}$ is the Lie algebra of H
  We have $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{h}$
- The Cartan decomposition is orthogonal with respect to the Killing form of $\mathfrak{g}$
- Complexification of **isotropy representation**

$$\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$$

A G-Higgs bundle on X

is a pair (E,φ) consisting of

- E a holomorphic principal $H^{\mathbb{C}}$-bundle over X
- φ a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}})\otimes K$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the associated vector bundle with fibre $\mathfrak{m}^{\mathbb{C}}$ via the complexified isotropy representation and K is the canonical line bundle of X

To define stability for G-Higgs bundles, consider for $s\in i\mathfrak{h}$:

- Parabolic subgroup $P_s=\{g\in H^{\mathbb{C}} \ : \ e^{ts}ge^{-ts} \text{ is bounded as } t\to\infty\}$
- Character $\chi_s:\mathfrak{p}_s\to\mathbb{C}$ defined by s ($\mathfrak{p}_s$ Lie algebra of P_s)
- Subspace $\mathfrak{m}_s=\{Y\in\mathfrak{m}^{\mathbb{C}} \ : \ \iota(e^{ts})Y \text{ is bounded as } t\to\infty\}$
- For σ a reduction of E to P_s

$$\deg(E)(\sigma,s):=\frac{i}{2\pi}\int_X\chi_s(F).$$

F: curvature of a connection on the P_s-bundle defined by σ

Stability of G-Higgs bundles

(E, φ) is:

- **(semi)stable** if

$$\deg(E)(\sigma, s) > 0 (\geq 0)$$

  for any $s \in i\mathfrak{h}$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$
- **polystable** if (E, φ) can be reduced to a G'-Higgs bundle, with $G' \subset G$ reductive and (E, φ) stable as a G'-Higgs bundle

The **moduli space of polystable G-Higgs bundles**

$$\mathcal{M}(X, G)$$

is the set of isomorphism classes of polystable G-Higgs bundles $\mathcal{M}(X, G)$ is as complex algebraic variety (GIT construction, **Schmitt**, 2008)

$G = \mathrm{SL}(n, \mathbb{C})$

- When G is a clasical group we can formulate the theory in terms of vector bundles
- In this case $H = \mathrm{SU}(n)$, $H^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$ and $\mathfrak{m}^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$
  Hence, an $\mathrm{SL}(n, \mathbb{C})$-**Higgs bundle** is equivalent to a pair (V, φ)
  V rank n holomorphic vector bundle with $\det V = \mathcal{O}$
  $\varphi : V \to V \otimes K$ with $\operatorname{Tr} \varphi = 0$
- (V, φ) is **stable**:
  $\deg(V') < 0$ for every $V' \subset V$ such that $\varphi(V') \subset V' \otimes K$
  (V, φ) is **polystable**:
  $(V, \varphi) = \oplus (V_i, \varphi_i)$ with $\deg V_i = 0$ and (V_i, φ_i) stable
- We recover the original notions introduced by **Hitchin** (1987)

$G = \mathrm{SU}(p, q)$

- In this case $H = \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$,
  $H^{\mathbb{C}} = \mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))$, and
  $\mathfrak{m}^{\mathbb{C}} = \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$
  Hence, an $\mathrm{SU}(p, q)$-**Higgs bundle** is equivalent to a tuple (V, W, β, γ)
  V and W are rank p and q holomorphic vector bundles, respectively, with $\det V \otimes \det W = \mathcal{O}$
  $\beta : W \to V \otimes K$ and $\gamma : V \to W \otimes K$
- (V, W, β, γ) is **stable**:
  $\deg(V') + \deg(W') < 0$ for every $V' \subset V$ and $W' \subset W$ such that $\beta(W') \subset V' \otimes K$ and $\gamma(V') \subset W' \otimes K$
  (V, W, β, γ) is **polystable** if the associated $\mathrm{SL}(p + q, \mathbb{C})$-Higgs bundle

$$V \oplus W \quad \text{and} \quad \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

  is polystable

$G = \mathrm{Sp}(2n, \mathbb{R})$

- In this case $H = \mathrm{U}(n)$, $H^{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$, and $\mathfrak{m}^{\mathbb{C}} = S^2(\mathbb{C}^n) \oplus S^2(\mathbb{C}^n)^*$
  Hence, an $\mathrm{Sp}(2n, \mathbb{R})$-**Higgs bundle** is equivalent to a triple (V, β, γ)
  V is rank n holomorphic vector bundles,
  $\beta : V^* \to V \otimes K$ and $\gamma : V \to V^* \otimes K$ **symmetric**
- stability of (V, β, γ) is a condition on certain two step filtrations $V_1 \subset V_2 \subset V$ (not enough to check the condition for subbundles of V)
  (V, β, γ) is **polystable** if the associated $\mathrm{SL}(p + q, \mathbb{C})$-Higgs bundle

$$V \oplus V^* \quad \text{and} \quad \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

  is polystable

Theorem

A G-Higgs (E, φ) is polystable if and only if there exists a reduction h of the structure group of E from $H^{\mathbb{C}}$ to H, such that

$$F_h - [\varphi, \tau_h(\varphi)] = 0 \quad \textbf{(Hitchin equation)}$$

- $\tau_h : \Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{m}^{\mathbb{C}}))$ is the combination of the anti-holomorphic involution in $E(\mathfrak{m}^{\mathbb{C}})$ defined by the compact real form at each point determined by h and the conjugation of 1-forms
- F_h is the curvature of the unique H-connection compatible with the holomorphic structure of E

Proved by:

- **Hitchin** (1987): $G = \mathrm{SL}(2, \mathbb{C})$
- **Simpson** (1988): G complex
- **Bradlow–G–Mundet** (2003) & **G–Gothen–Mundet** (2009): G real

Non-abelian Hodge correspondence

Let S be a smooth compact surface and J be a complex structure on S. Let $X = (S, J)$. There is a homeomorphism

$$\mathcal{R}(S, G) \cong \mathcal{M}(X, G)$$

- Let (E, φ) be a polystable G-Higgs bundle and h a solution to Hitchin equations

$$\nabla = \bar{\partial}_E - \tau_h(\bar{\partial}_E) + \varphi - \tau_h(\varphi)$$

  is a flat G-connection and the holonomy representation ρ is reductive
- **Converse**: Existence of a **harmonic metric** on a reductive flat G-bundle. Proved by **Donaldson** (1987) for $G = \mathrm{SL}(2, \mathbb{C})$ and **Corlette** (1988) for real reductive G

Theorem

Let ρ be a representation of $\pi_1(X)$ in G with corresponding flat G-bundle E_ρ. Let $E_\rho(G/H)$ be the associated G/H-bundle. Then the existence of a harmonic section of $E_\rho(G/H)$ is equivalent to the reductiveness of ρ.

- Recall that a section of $E_\rho(G/H) \to X$ is equivalent to a $\pi_1(X)$-equivariant map

$$f : \tilde{X} \to G/H$$

where $\pi_1(X)$ acts on $\tilde{X}$ by Deck transformations and on G/H via ρ
- The harmonicity of the section is equivalent to solving Hitchin equations

3. Topological invariants

- Given $\rho\colon \pi_1(S) \to G$, there is an associated flat G-bundle on S, defined as $E_\rho = \widetilde{S} \times_\rho G$ ($\widetilde{S}$: universal cover of S):

  $\mathrm{Hom}(\pi_1(S), G)/G \cong H^1(S, G) =$ iso. classes of flat G-bundles

- Let $\widetilde{G}$ be the universal covering group of G. We have an exact sequence

$$1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$$

  which gives rise to the (pointed sets) cohomology sequence

$$H^1(S, \widetilde{G}) \to H^1(S, G) \xrightarrow{c} H^2(S, \pi_1(G))$$

- **topological invariant** of ρ:
  $c(\rho) := c(E_\rho) \in H^2(X, \pi_1(G)) \cong \pi_1(G)$
- We can define the subvariety

$$\mathcal{R}_c(S, G) := \{\rho \in \mathcal{R}(S, G) \ : \ c(\rho) = c\}$$

- Similarly, we can define a topological invariant of a G-Higgs bundle (E, φ) over X as the topological class of the $H^{\mathbb{C}}$-bundle E (recall $H \subset G$ is a maximal compact subgroup)
- $H^1(X, \underline{H^{\mathbb{C}}}) =$ isomorphisms classes of $H^{\mathbb{C}}$-bundles
  We have

$$H^1(X, \underline{\widetilde{H}^{\mathbb{C}}}) \to H^1(X, \underline{H^{\mathbb{C}}}) \xrightarrow{c} H^2(X, \pi_1(H^{\mathbb{C}}))$$

- **topological invariant** of (E, φ):

$$c(E, \varphi) \in H^2(X, \pi_1(H^{\mathbb{C}})) \cong \pi_1(H^{\mathbb{C}})$$

- We can define the subvariety

$$\mathcal{M}_c(X, G) := \{(E, \varphi) \in \mathcal{M}(X, G) \ : \ c(E, \varphi) = c\}$$

- Recall $\pi_1(G) \cong \pi_1(H) \cong \pi_1(H^{\mathbb{C}})$
- For $c \in \pi_1(G) \cong \pi_1(H^{\mathbb{C}})$ we have de homeomorphism

$$\mathcal{R}_c(S, G) \cong \mathcal{M}_c(X, G)$$

Theorem

If G is compact (Ramanathan, 1975) or complex (J. Li, 1993; G-Oliveira, 2017)

$$\pi_0(\mathcal{R}(S, G)) = \pi_0(\mathcal{M}(X, G)) \cong \pi_1(G)$$

- The story is very different for **non-compact** real Lie groups (non-complex). The map

$$\pi_0(\mathcal{R}(S, G)) = \pi_0(\mathcal{M}(X, G)) \to \pi_1(G)$$

may be neither surjective nor injective!

4. $G = \mathrm{SL}(2,\mathbb{R})$

- The topological invariant of $\rho \in \mathcal{R}(S, \mathrm{SL}(2,\mathbb{R}))$ in this case is an integer (basically the **Euler class**) $d \in \mathbb{Z} \cong \pi_1(G)$
- $$\mathcal{R}_d := \{\rho \in \mathcal{R}(S, \mathrm{SL}(2,\mathbb{R})) \quad : \quad \text{with Euler class } \; d\}$$

Theorem (Milnor, 1958)

$\mathcal{R}_d$ is empty unless
$$|d| \leq g - 1$$

- An $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle is a tuple (L, β, γ)
L line bundle over X $\quad \beta \in H^0(X, L^2K)$ and $\gamma \in H^0(X, L^{-2}K)$
Equivalently it can be seen as an $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle (V, φ) with $V = L \oplus L^{-1}$ and
$$\varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$
- Milnor's inequality follows from the stability of (V, φ) (**Hitchin**, 1987)

Theorem (Goldman, 1988; Hitchin 1987)

- $\mathcal{R}_d$ is connected if $|d| < g - 1$
- $\mathcal{R}_d$ has 2^{2g} connected components if $|d| = g - 1$

- Let $\mathcal{R}_{\max} := \mathcal{R}_d$ for $|d| = g - 1$
- Each connected component of $\mathcal{R}_{\max}$ consists entirely of **Fuchsian representations** (discrete and faithful) and can be identified with **Teichmüller space** (**Goldman**, 1980)
- **Question**: Are there other (non compact) groups with similar features to those of $\mathrm{SL}(2,\mathbb{R})$?
- **Split** real groups ($\mathrm{SL}(2,\mathbb{R})$ is the split real form of $\mathrm{SL}(2,\mathbb{C})$)
- Groups of **Hermitian type** (Poincaré disc $D = \mathrm{SL}(2,\mathbb{R})/SO(2)$)
- **Orthogonal groups** $\mathrm{SO}(p,q)$ ($\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}_0(1,2)$)

5. Split real forms

- **Split** real form = in the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, the space $\mathfrak{m}$ contains a maximal abelian subalgebra of $\mathfrak{g}$
- Every complex semisimple Lie group has a split real form
  Examples: $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}(n, n)$, $\mathrm{SO}(n, n+1)$
- Consider $G = \mathrm{SL}(n, \mathbb{R})$
  A basis for the **invariant polynomials** on $\mathfrak{sl}(n, \mathbb{C})$ is provided by the coefficients of the characteristic polynomial of a trace-free matrix,

$$\det(x - A) = x^n + p_1(A)x^{n-2} + \ldots + p_{n-1}(A),$$

  where $\deg(p_i) = i + 1$.

- Consider the **Hitchin map**

$$p : \mathcal{M}(X, \mathrm{SL}(n,\mathbb{C})) \to \bigoplus_{i=1}^{n-1} H^0(K^{i+1})$$

  defined by

$$p(E,\varphi) = (p_1(\varphi), \dots, p_{n-1}(\varphi)),$$

- **Hitchin** (1992) constructed a **section** of this map giving an isomorphism between the vector space $\bigoplus_{i=1}^{n-1} H^0(K^{i+1})$ and a **connected component** of the moduli space $\mathcal{M}(X, \mathrm{SL}(n,\mathbb{R})) \subset \mathcal{M}(X, \mathrm{SL}(n,\mathbb{C}))$
- This is called a **Hitchin component** (coincides with a Teichmüller component $\cong H^0(X, K^2)$ when $G = \mathrm{SL}(2,\mathbb{R})$)
- Hitchin (1992) gives a general construction for any split real form

- The Hitchin component is essentially **unique** if G is a split form of **adjoint type** (i.e. without centre)
- Every representation in the Hitchin component can be deformed to a representation factoring as $\pi_1(S) \to \mathrm{SL}(2,\mathbb{R}) \to G$, where $\pi_1(S) \to \mathrm{SL}(2,\mathbb{R})$ is in a Teichmüller component and $\mathrm{SL}(2,\mathbb{R}) \to G$ is a **principal representation**
- A Hitchin component consists entirely of **discrete and faithful** representations (**Labourie** 2006)
- For $G = \mathrm{SL}(3,\mathbb{R})$, the Hitchin component parametrizes **convex projective structures** on the compact surface (**Choi–Goldman** 1997)
- The Hitchin component parameterizes certain type of **geometric structures** (**Guichard–Wienhard** 2008)

6. The Hitchin map and the Hitchin–Kostant–Rallis section

- G semisimple real Lie group
- $H \subset G$ maximal compact subgroup of G
- θ Cartan involution of $\mathfrak{g}$, Lie algebra of G, defining the **Cartan decomposition**:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$$

- $\mathfrak{a} \subset \mathfrak{m}$ **maximal abelian subalgebra**
-

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{\lambda \in \Lambda(\mathfrak{a}^{\mathbb{C}})} \mathfrak{g}_{\lambda},$$

  $\Lambda(\mathfrak{a}^{\mathbb{C}}) \subset (\mathfrak{a}^{\mathbb{C}})^* =$ **restricted roots** of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{a}^{\mathbb{C}}$
- $W(\mathfrak{a}^{\mathbb{C}})$: **restricted Weyl group** of $\mathfrak{g}^{\mathbb{C}}$ associated to $\mathfrak{a}^{\mathbb{C}}$ (group of automorphisms of $\mathfrak{a}^{\mathbb{C}}$ generated by reflections on the hyperplanes defined by the restricted roots $\Lambda(\mathfrak{a}^{\mathbb{C}})$)

- **Chevalley**:

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}} \cong \mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})}$$

  Hence we have an algebraic morphism

$$\mathfrak{m}^{\mathbb{C}} \twoheadrightarrow \mathfrak{m}^{\mathbb{C}} /\!\!/ H^{\mathbb{C}} \cong \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$$

- This map is $\mathbb{C}^*$-equivariant. In particular, it induces a morphism

$$h : \mathfrak{m}^{\mathbb{C}} \otimes K \to \mathfrak{a}^{\mathbb{C}} \otimes K/W(\mathfrak{a}^{\mathbb{C}}).$$

- The map is also $H^{\mathbb{C}}$-equivariant, thus defining a morphism

$$h : \mathcal{M}(X,G) \to B(X,G) := H^0(X, \mathfrak{a}^{\mathbb{C}} \otimes K/W(\mathfrak{a}^{\mathbb{C}}))$$

- h is called the **Hitchin map**
- $B(X,G)$ is called the **Hitchin base**

- **Kostant–Rallis** constructed $\widehat{\mathfrak{g}} \subset \mathfrak{g}$: **maximal split subalgebra** of $\mathfrak{g}$
- $\widehat{\mathfrak{g}}$ is a θ-invariant subalgebra of $\mathfrak{g}$ generated by $\mathfrak{a}$ and a principal normal three dimensional subalgebra ($\cong \mathfrak{sl}(2,\mathbb{C})$) of $\mathfrak{g}^{\mathbb{C}}$ invariant by the conjugation defining $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$
- $\widehat{G} \subset G$: **maximal split subgroup** (analytic subgroup corresponding to $\widehat{\mathfrak{g}}$)
- $\widehat{H} \subset \widehat{G}$ maximal compact subgroup, $\widehat{\mathfrak{g}} = \widehat{\mathfrak{h}} + \widehat{\mathfrak{m}}$

  We have

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}} \cong \mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})} \cong \mathbb{C}[\widehat{\mathfrak{m}}^{\mathbb{C}}]^{\widehat{H}^{\mathbb{C}}}.$$

  Hence

$$B(X,G) = B(X,\widehat{G})$$

- If G is complex $\widehat{G} = G_{\text{split}}$, the split real form of G. In particular $B(X,G) = B(X,G_{\text{split}})$

Let G be a real form of a complex semisimple Lie group $G^{\mathbb{C}}$:

- G is **quasi-split** if $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{a})$ is abelian (split real forms, $\mathrm{SU}(p,p)$, $\mathrm{SU}(p,p+1)$, $\mathrm{SO}(p,p+2)$, and E_6^2)
- $x \in \mathfrak{m}^{\mathbb{C}}$ is said to be **regular** if $\dim \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x) = \dim \mathfrak{a}^{\mathbb{C}}$, where $\mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x) = \{y \in \mathfrak{m}^{\mathbb{C}} \;:\; [y,x]=0\}$.
- G is quasi-split if and only if $\mathfrak{m}^{\mathbb{C}} \cap \mathfrak{g}^{\mathbb{C}}_{\mathrm{reg}} = \mathfrak{m}^{\mathbb{C}}_{\mathrm{reg}}$.
- There is an exact sequence

$$1 \to G \to N_{G^{\mathbb{C}}}(G) \to Q \to 1,$$

  where Q is finite

Theorem (G-Peón–Ramanan 2017)

The choice of a square root of K determines $|Q|$ inequivalent sections of the Hitchin map

$$h : \mathcal{M}(X, G) \to B(X, G).$$

Each such section s_G satisfies

1. If G is quasi-split, $s_G(B(X, G))$ is contained in the stable locus of $\mathcal{M}(X, G)$.
2. If G is not quasi-split, the image of the section is contained in the strictly polystable locus.
3. For arbitrary groups, the Higgs field is everywhere regular.
4. The section factors through $\mathcal{M}(X, \widehat{G})$.
5. If $G = G_{\mathrm{split}}$ is the split real form s_G is the factorization of the Hitchin section through $\mathcal{M}(X, G_{\mathrm{split}})$ and $s_G(B(X, G_{\mathrm{split}}))$ is a topological component of $\mathcal{M}(X, G_{\mathrm{split}})$ (**Hitchin component**).

7. Non-compact real forms of Hermitian type

- G of **Hermitian type** means that G/H admits a complex structure compatible with the Riemannian structure of G/H, making G/H a Kähler manifold
- If G is simple the centre of $\mathfrak{h}$ is one-dimensional and the almost complex structure on G/H is defined by a generating element in $J \in Z(\mathfrak{h})$
- This complex structure defines a decomposition

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{+} \oplus \mathfrak{m}^{-},$$

where $\mathfrak{m}^{+}$ and $\mathfrak{m}^{-}$ are the $(1,0)$ and the $(0,1)$ part of $\mathfrak{m}^{\mathbb{C}}$ respectively

- **Classical** connected simple groups of Hermitian type: $\mathrm{SU}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}^*(2n)$, $\mathrm{SO}_0(2,n)$
- Two **exceptional** real forms E_6^{-14} and E_7^{-25}

- Let (E, φ) be a G-Higgs bundle over X. The decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ gives a vector bundle decomposition

$$E(\mathfrak{m}^{\mathbb{C}}) = E(\mathfrak{m}^+) \oplus E(\mathfrak{m}^-)$$

- Hence

$$\varphi = (\varphi^+, \varphi^-) \in H^0(X, E(\mathfrak{m}^+) \otimes K) \oplus H^0(X, E(\mathfrak{m}^-) \otimes K)$$

- The torsion-free part of $\pi_1(H)$ is isomorphic to $\mathbb{Z}$ (most of the time $\pi_1(H) \cong \mathbb{Z}$) and hence the topological invariant of either a representation of $\pi_1(X)$ in G, or of a G-Higgs bundle, is essentialy given by an **integer** $d \in \mathbb{Z}$. This is related to the so-called Toledo invariant.

- The **Toledo character** $\chi_T : \mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$ is defined, for $Y \in \mathfrak{h}^{\mathbb{C}}$ in terms of the Killing form, by

$$\chi_T(Y) = \frac{1}{N}\langle -iJ, Y\rangle,$$

where N is the dual Coxeter number.

- The Toledo character χ_T defines a symmetric Kähler form on G/H by

$$\omega(Y, Z) = i\chi_T([Y, Z]), \text{ for } Y, Z \in \mathfrak{m},$$

with minimal holomorphic sectional curvature -1.

- Define o_J to be the order of $e^{2\pi J}$ and $\ell = |Z_0^{\mathbb{C}} \cap [H^{\mathbb{C}}, H^{\mathbb{C}}]|$, where $Z_0^{\mathbb{C}}$ is the identity component of $Z(H^{\mathbb{C}})$. For $q \in \mathbb{Q}$, the character $q\chi_T$ lifts to a character $H^{\mathbb{C}} \to \mathbb{C}^*$ if and only if q is an integral multiple of

$$q_T = \frac{\ell N}{o_J \dim \mathfrak{m}}$$

- Let (E,φ) be a G-Higgs bundle. Maybe up to an integer multiple, χ_T lifts to a character $\tilde{\chi}_T$ of $H^{\mathbb{C}}$. Let $E(\tilde{\chi}_T)$ be the line bundle associated to E via the character $\tilde{\chi}_T$. We define the **Toledo invariant** τ of (E,φ)

$$\tau = \tau(E) := \deg(E(\tilde{\chi}_T)).$$

  If $\tilde{\chi}_T$ is not defined, but only $\tilde{\chi}_T^q$, one must replace the definition by $\frac{1}{q}\deg E(\tilde{\chi}_T^q)$.
- Let $\rho : \pi_1(X) \to G$ be reductive and let (E,φ) be the corresponding polystable G-Higgs bundle. Let $f : \tilde{X} \to G/H$ be the corresponding harmonic metric. Then

$$\tau(E) = \frac{1}{2\pi}\int_X f^*\omega,$$

  In particular, $\tau(E) = \tau(\rho)$ — the Toledo invariant of ρ.
- Let $d \in \mathbb{Z}$ be the projection on the torsion-free part of the class $c(E,\varphi) = c(\rho)$. Then d is related to τ by

$$\tau = \frac{d}{q_T}$$

Theorem (Milnor–Wood inequality)

Let (E, φ) be semistable. The Toledo invariant τ satisfies

$$|\tau| \leq \mathrm{rank}(G/H)(2g-2)$$

- Proved for the classical groups for representations by **Domic–Toledo** (1987) and for Higgs bundles by **Bradlow–G–Gothen** (2001)
- General proof for representations by **Burger–Iozzi–Wienhard** (2010), and for Higgs bundles by **Biquard–G–Rubio** (2017)
- For $G = \mathrm{SU}(p,q)$, G-Higgs bundle: (V, W, β, γ)
  Toledo invariant $= \tau = 2 \deg V$ $(\deg W = -\deg V)$
  Milnor–Wood inequality: $|\tau| \leq \min\{p,q\}(2g-2)$
- For $G = \mathrm{Sp}(2n, \mathbb{R})$, G-Higgs bundle: (V, β, γ)
  Toledo invariant $= \tau = 2 \deg V$
  Milnor–Wood inequality: $|\tau| \leq n(2g-2)$

- G/H can be realized as a **bounded symmetric domain** $\mathcal{D}$ in $\mathfrak{m}^+$, say (**Cartan** for the classical groups and **Harish-Chandra** in general)
- $\mathcal{D}$ is called of **tube type** if it is biholomorphic to a tube T_Ω over a cone Ω: $T_\Omega = \mathbb{R}^n + i\Omega$, for $\Omega \subset \mathbb{R}^n$
- The **Poincaré disc**, the domain corresponding to $G = \mathrm{SU}(1,1)$, is of tube type. The tube is the upper-half plane and the biholomorphism is the **Cayley transform**
- The **Shilov boundary** of $\mathcal{D}$ is defined as the smallest closed subset $\check{S}$ of the topological boundary $\partial\mathcal{D}$ for which every function f continuous on $\overline{\mathcal{D}}$ and holomorphic on $\mathcal{D}$ satisfies that

$$|f(z)| \leq \max_{w\in\check{S}} |f(w)| \quad \text{for every} \quad z \in \mathcal{D}$$

- $\mathcal{D}$ is of **tube type** if and only if $\check{S}$ is a **compact symmetric space** of the form H/H'. In this case $\Omega = G'/H'$ is its **non-compact dual symmetric space**

- The symmetric spaces defined by $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}_0(2, n)$ are of tube type.
- The symmetric space defined by $\mathrm{SU}(p, q)$ is of tube type if and only if $p = q$.
- The symmetric space defined by $\mathrm{SO}^*(2n)$ is of tube type if and only if n is even.
- The E_7 Hermitian real form is of tube type
- The E_6 Hermitian real form is **not** of tube type
- Every bounded symmetric domain has a maximal tube subdomain

- Want to study the **Maximal Toledo invariant** moduli space in the tube case (the non-tube reduces to the tube case)

$$\mathcal{M}_{\max}(X,G) := \mathcal{M}_\tau(X,G) \ \text{ for } \ |\tau| = \operatorname{rank}(G/H)(2g-2)$$

Assume that G is of tube type

- From Chevally theorem on invariant polynomials, one can deduce the existence of a homogeneous polynomial function $\det : \mathfrak{m}^+ \to \mathbb{C}$, satisfying for $h \in H^{\mathbb{C}}$ and $x \in \mathfrak{m}^+$

$$\det(\operatorname{Ad}(h)x) = \widetilde{\chi}_T(h)\det(x)$$

- Define

$$\mathfrak{m}^+_{\text{reg}} := \{x \in \mathfrak{m}^+ \quad \det(x) \neq 0\}$$

Then

$$\mathfrak{m}^+_{\text{reg}} \cong H^{\mathbb{C}}/H'^{\mathbb{C}}$$

(recall that $\check{S} = H/H'$)

- Let G'/H' be the non-compact dual of the Shilov boundary H/H', and $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$ be the corresponding Cartan decomposition. Any element $x \in \mathfrak{m}^+_{\mathrm{reg}}$ defines an $H'^{\mathbb{C}}$-equivariant isomorphism

$$\mathrm{ad}(x) : \mathfrak{m}^- \to \mathfrak{m}'^{\mathbb{C}}$$

- Let $(E, \varphi^+, \varphi^-) \in \mathcal{M}_{\max}(X, G)$ with $\tau = -\operatorname{rank}(G/H)(2g-2)$. Then $\varphi^+(x) \in \mathfrak{m}^+_{\mathrm{reg}} \cong H^{\mathbb{C}}/H'^{\mathbb{C}}$ for every $x \in X$ and if κ is an o_J-root of K, φ^+ defines a reduction of the $H^{\mathbb{C}}$-bundle $E \otimes \kappa$ to an $H'^{\mathbb{C}}$-bundle E'.
- We also have an isomorphism

$$\mathrm{ad}(\varphi^+) : E'(\mathfrak{m}^-) \to E'(\mathfrak{m}'^{\mathbb{C}})$$

So that we can define a Higgs field

$$\varphi' := \mathrm{ad}(\varphi^+)(\varphi^-) = [\varphi^+, \varphi^-] \in H^0(X, E'(\mathfrak{m}'^{\mathbb{C}}) \otimes K^2).$$

The pair (E', φ') is a K^2-twisted G'-Higgs bundle.

Theorem (Cayley Correspondence)

Let G be a such G/H is a Hermitian symmetric space of tube type, and let $\Omega = G'/H'$ be the non-compact dual of the Shilov boundary $\check{S} = H/H'$ of G/H. Assume that o_J divides $2g-2$. Then the correspondence $(E, \varphi) \mapsto (E', \varphi')$ defines an isomorphism

$$\mathcal{M}_{\max}(X, G) \cong \mathcal{M}_{K^2}(X, G'),$$

where $\mathcal{M}_{K^2}(X, G')$ is the moduli space of K^2-twisted G'-Higgs bundles

- Proved for the classical groups by **Bradlow–G–Gothen** (2006) **G–Gothen–Mundet** (2013) ($G = \mathrm{Sp}(2n, \mathbb{R})$)
- General case proved by **Biquard–G–Rubio** (2017)

- The connected components of $\mathcal{M}(X, G)$ are not fully distinguished by the usual topological invariants. The dual group G' detects **new hidden invariants** (for example for $G = \mathrm{Sp}(2n, \mathbb{R})$, $\check{S} = H/H'$, $G' = \mathrm{GL}(n, \mathbb{R})$, — **Stiefel–Whitney classes**
- $\mathcal{R}_{\max}(S, G)$ consists entirely of discrete and faithful representations (**Burger–Iozzi–Labourie–Wienhard**, 2006)
- The mapping class group of S acts properly on $\mathcal{R}_{\max}(S, G)$ (**Wienhard**, 2006)
- All common features with **Hitchin components**
- $G = \mathrm{Sp}(2n, \mathbb{R})$ is **both split and Hermitian** and

$$\mathcal{M}_{\mathrm{Hitchin}}(X, G) \subset \mathcal{M}_{\max}(X, G)$$

- A **higher Teichmüller component** of $\mathcal{R}(S, G)$ or $\mathcal{M}(X, G)$ is defined as one that has this kind of properties
- **Question:** Are there other groups besides **split** and **hermitian** real forms for which **higher Teichmüller components** exist?

8. Higher Teichmüller components for $G = \mathrm{SO}(p,q)$

- Joint work with **M. Aparicio, S. Bradlow, B. Collier, P. Gothen and A. Oliveira** (arXiv:1801.08561 and arxiv:1802.08093)
- An $\mathrm{SO}(p,q)$-**Higgs bundle** is defined by a triple (V, W, η) where V and W are respectively rank p and rank q vector bundles with orthogonal structures such that $\det(W) \simeq \det(V)$, and η is a holomorphic bundle map $\eta : W \to V \otimes K$
- For $p > 2$, rank p orthogonal bundles on X are classified topologically by their **first and second Stiefel–Whitney classes**, $sw_1 \in H^1(X, \mathbb{Z}_2)$ and $sw_2 \in H^2(X, \mathbb{Z}_2)$
- Since $\det(W) \simeq \det(V)$ $sw_1(V) = sw_1(W)$
  The components of the moduli space $\mathcal{M}(\mathrm{SO}(p,q))$ are thus **partially** labeled by triples $(a, b, c) \in \mathbb{Z}_2^{2g} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $a = sw_1(V) \in H^1(X, \mathbb{Z}_2)$, $b = sw_2(V) \in H^2(X, \mathbb{Z}_2)$, and $c = sw_2(W) \in H^2(X, \mathbb{Z}_2)$

$$\mathcal{M}(\mathrm{SO}(p,q)) = \coprod \mathcal{M}^{a,b,c}(\mathrm{SO}(p,q))$$

Proposition

Assume that $2 < p \leq q$. For every $(a, b, c) \in \mathbb{Z}_2^{2g} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ the space $\mathcal{M}^{a,b,c}(\mathrm{SO}(p, q))$ has a non-empty connected component denoted by $\mathcal{M}_{\mathrm{top}}^{a,b,c}(\mathrm{SO}(p, q))$

- Define

$$\mathcal{M}_{\mathrm{top}}(\mathrm{SO}(p, q)) = \coprod_{a,b,c} \mathcal{M}_{\mathrm{top}}^{a,b,c}(\mathrm{SO}(p, q))$$

- Our main result shows that the moduli space $\mathcal{M}(\mathrm{SO}(p, q))$ has additional **exotic** components disjoint from the components of $\mathcal{M}_{\mathrm{top}}(\mathrm{SO}(p, q))$
- We identify these exotic components as products of moduli spaces of L-**twisted Higgs bundles**, where in each factor L is a positive power of the canonical bundle K

Theorem (Generalized Cayley Correspondence)

Fix integers (p, q) such that $2 < p < q - 1$. For each choice of $a \in \mathbb{Z}_2^{2g}$ and $c \in \mathbb{Z}_2$, the moduli space $\mathcal{M}(\mathrm{SO}(p,q))$ has a connected component disjoint from $\mathcal{M}_{\mathrm{top}}(\mathrm{SO}(p,q))$. This component is isomorphic to

$$\mathcal{M}^{a,c}_{K^p}(\mathrm{SO}(1,q-p+1)) \times \mathcal{M}_{K^2}(\mathrm{SO}_0(1,1)) \times \cdots \times \mathcal{M}_{K^{2p-2}}(\mathrm{SO}_0(1,1))$$

- The case of $\mathrm{SO}(2,q)$ is special since the group is of Hermitian type
- The existence of exotic components for $\mathrm{SO}(p,p+1)$ for $p > 2$ is proved by **Collier** (2017)
- **Conjecture**: These exotic components are higher Teichmüller components (i.e. consist entirely of discrete and faithful representations)

- **Evidence**: Notion of **positivity** recently introduced by **Guichard–Wienhard** (2016)
- This generalizes **Lusztig positivity** for split real forms (related to **Fock–Goncharov positivity**)
- It generalizes also a notion of positivity for Hermitian groups (related to **causality**)
- **Conjecture** (Guichard–Wienhard): The only groups for which there are higher Teichmüller components are those admitting a **positive structure**
- The **only** classical groups admitting positive structures are: **split groups, hermitian groups of tube type** and $\mathrm{SO}(p,q)$!
- $\mathrm{Sp}(2n,\mathbb{R})$ admits two different positive structures
- The exceptional groups admitting positive structures are real forms of F_4, E_6, E_7 and E_8, whose restricted root system is of type F_4
- F_4^4 is such an example: Existence of exotic components is proved by **Bradlow–Collier–G–Gothen–Oliveira** (work in progress). This group has also two different positive structures

9. Prehomogeneous vector spaces

Theory introduced by **Sato** (**Sato–Kimura**, 1977)

G Complex reductive Lie group

- A **prehomogeneous vector space** for G is a complex finite dimensional vector space V together with a holomorphic representation of G on V such that G has an open and dense orbit in V. It turns out that such an orbit is unique and dense.
- Let V be a prehomogeneous vector space for G with representation ρ. A non-constant holomorphic function $f : V \to \mathbb{C}$ is called a **relative invariant** for the action of G if there exists a **character** $\chi : G \to \mathbb{C}^*$ such that

$$f(\rho(g)x) = \chi(g)f(x) \quad \text{for every} \quad g \in G \quad \text{and} \quad x \in V.$$

- Let Ω be the open orbit in V and $S := V - \Omega$ be the singular set in V. The prehomogeneous vector space V is called **regular** if the isotropy subgroup G^x is reductive for any $x \in \Omega$.

- A relative invariant is, up to constant multiple, uniquely determined by its corresponding character. In particular any relative invariant is a homogeneous function.
- The existence of a relative invariant is equivalent to S having an irreducible component of codimension one.
- The prehomogeneous vector space V is regular if and only if S is a hypersurface.
- Let $S = \cup_{i=1}^{i=l} S_i$ be the decomposition of S in irreducible components. If V is regular, then for every S_i (which is a hypersurface) there is a character χ_i and a corresponding relative invariant f_i such that $S_i = \{x \in V \mid f_i(x) = 0\}$. The number l coincides with the number of G-irreducible summands in V. In particular, if V is irreducible $l = 1$.

Assume that V is regular

- Let $x \in \Omega$. Let χ be a character of G and f be the corresponding relative invariant. We define the **rank of** x **with respect to** χ as

$$\mathrm{rank}_\chi(x) = \deg(f).$$

- We can actually define the rank of any point $x \in V$. To do this, we polarize f to get an r-linear map q on V such that $q(x, \ldots, x) = f(x)$; then the rank of x is the maximal integer r' such that the $(r - r')$-form $q(x, \ldots, x, \cdot, \ldots, \cdot)$ is not identically zero.
- In fact, rank can be defined for any point of V even if V is not regular

- Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A $\mathbb{Z}$-grading of $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$$

  such that $[\mathfrak{g}_j, \mathfrak{g}_k] = \mathfrak{g}_{j+k}$.

- A $\mathbb{Z}$-grading of $\mathfrak{g}$ defines a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$, given by

$$\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$$

  And in fact any $\mathbb{Z}$-grading is defined by a parabolic subalgebra.

- Let G be a complex semisimple Lie group with a graded Lie algebra $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$, and let G_0 be the analytic subgroup of G with Lie algebra $\mathfrak{g}_0$. Then $\mathfrak{g}_1$ is a prehomogeneous space for the adjoint action of G_0. In fact $\mathfrak{g}_1$ has only a finite number of G_0-orbits (hence one of them must be open).
  The prehomogeneus vector space $\mathfrak{g}_1$ is said to be of **parabolic type**.

Example: The Hermitian case

- The representation of $H^{\mathbb{C}}$ on $\mathfrak{m}^+$, where G/H is Hermitian is a prehomogeneous vector space of parabolic type:
  $\mathbb{Z}$-grading:
  $$\mathfrak{g}^{\mathbb{C}} = \mathfrak{m}^- \oplus \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^+$$
- $\mathfrak{m}^+$ is regular if and only if G/H is of tube type
- If G is simple and of tube type, then $\mathfrak{m}^+$ is irreducible and det is the basic relative invariant for the action of $H^{\mathbb{C}}$ on $\mathfrak{m}^+$, corresponding of course to the Toledo character of $H^{\mathbb{C}}$
- The same applies to $\mathfrak{m}^-$

- Let G be a real form of a semisimple complex Lie group $G^{\mathbb{C}}$, i.e. $G = (G^{\mathbb{C}})^{\sigma}$, where σ is a conjugation of $G^{\mathbb{C}}$. The group G is said to be of **Hodge type** if $\sigma = \tau\,\mathrm{Int}(g)$, where τ is a compact conjugation and $g \in G^{\mathbb{C}}$
- Let G be Lie group of Hodge type and $H \subset G$ be a maximal compact subgroup. A **period domain** is a G-homogeneous space G/H_0 equipped with a homogeneous complex structure, where $H_0 \subset H$ is the H-centralizer of a compact torus $T_0 \subset H$
- A period domain defines a canonical $\mathbb{Z}$-grading $\mathfrak{g}^{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k^{\mathbb{C}}$, satifying $H_0^{\mathbb{C}}$ is the complex subgroup of $G^{\mathbb{C}}$ corresponding to $\mathfrak{g}_0^{\mathbb{C}}$, and

$$\mathfrak{h}^{\mathbb{C}} = \bigoplus_{k=even} \mathfrak{g}_k^{\mathbb{C}} \quad \mathfrak{m}^{\mathbb{C}} = \bigoplus_{k=odd} \mathfrak{g}_k^{\mathbb{C}}.$$

We will refer to the structure defined above as a **Hodge structure** on G.

Some examples of groups of Hodge type and Hodge structures

- Hermitian groups are of Hodge type. The symmetric space G/H is itself a period domain ($H_0 = H$)
- $G = \mathrm{SO}(2p, q)$, $H = \mathrm{SO}(2p) \times \mathrm{SO}(q)$, $H_0 = \mathrm{U}(p) \times \mathrm{SO}(q)$
- $G = \mathrm{Sp}(p, q)$, $H = \mathrm{Sp}(p) \times \mathrm{Sp}(q)$, $H_0 = \mathrm{U}(p) \times \mathrm{Sp}(q)$
- Let $G^{\mathbb{C}}$ be a semisimple complex Lie groups and $\mathfrak{g}^{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k^{\mathbb{C}}$ be a $\mathbb{Z}$-grading. Then there exists a real form G of $G^{\mathbb{C}}$ of Hodge type and a Hodge structure on G such that $\mathfrak{g}_{2i}^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{g}_{2i+1}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$
- Let's go back to Higgs bundles!

10. Hodges bundles and variations of Hodge structure

G: real form of a semisimple complex Lie group $G^{\mathbb{C}}$

- Consider a $\mathbb{Z}$-grading of $\mathfrak{g}^{\mathbb{C}}$

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k^{\mathbb{C}}, \quad \mathfrak{g}_k^{\mathbb{C}} = \mathfrak{h}_k^{\mathbb{C}} \oplus \mathfrak{m}_k^{\mathbb{C}}$$

  A G-Higgs bundle (E,φ) in $\mathcal{M}(X,G)$ is called a **Hodge bundle** if E admits a reduction of structure group to $H_0^{\mathbb{C}}$, the Lie group corresponding to $\mathfrak{h}_0^{\mathbb{C}}$, and $\varphi \in H^0(X, E(\mathfrak{m}_1^{\mathbb{C}}) \otimes K)$
- Consider the natural action of $\mathbb{C}^*$ on the moduli space $\mathcal{M}(X,G)$ of G-Higgs bundles given for $\lambda \in \mathbb{C}^*$ by

$$\lambda \cdot (E,\varphi) := (E, \lambda\varphi)$$

  A point (E,φ) in $\mathcal{M}(X,G)$ is fixed under the $\mathbb{C}^*$-action if and only if it is a Hodge bundle (**Simpson**, 1992)
- There is a good reason to call this a Hodge bundle: if (E,φ) is a Hodge bundle then there exists a real form of Hodge type $G' \subset G^{\mathbb{C}}$ such that the grading of $\mathfrak{g}^{\mathbb{C}}$ comes from a **period domain** G'/H_0

- Assume that G is of Hodge type anf let (E, φ) in $\mathcal{M}(X, G)$. Let $\rho : \pi_1(X) \to G$ be the corresponding representation. Then (E, φ) is a Hodge bundle coming from a Hodge structure on G if and only if the equivariant harmonic map $f : \tilde{X} \to G/H$ lifts to an equivariant **horizontal holomorphic map** $\tilde{f} : \tilde{X} \to G/H_0$
The pair $(\rho, \tilde{f})$ is called a **variation of Hodge structure** (VHS)
- Solving Hitchin equations for a polystable G-Higgs bundle one obtains a function

$$f : \mathcal{M}(X, G) \to \mathbb{R}$$

defined by $f(E, \varphi) = \|\varphi\|^2_{L^2}$, called **Hitchin function**
- A smooth point (E, φ) is a **critical point** of the Hitchin function if and only it is a **Hodge bundle**

- Assume that G is of Hodge type and G is equipped with a Hodge structure: $H_0 \subset H$, $\mathbb{Z}$-grading of $\mathfrak{g}^{\mathbb{C}}$, etc. Then $\mathfrak{g}_1^{\mathbb{C}}$ is a prehomogeneous space for the action of $H_0^{\mathbb{C}}$
Assume that $\mathfrak{g}_1^{\mathbb{C}}$ is **regular**

Milnor–Arakelov inequality (Biquard–Collier–G–Toledo, 2018)

Let (E, φ) be a semistable Hodge bundle. Let χ be a character of $H_0^{\mathbb{C}}$. Then

$$|\tau_\chi| \leq \mathrm{rank}_\chi(\varphi)(2g-2),$$

where $\mathrm{rank}_\chi(\varphi)$ is the rank of $\varphi(x)$ for a generic $x \in X$. In particular

$$|\tau_\chi| \leq r_\chi(2g-2),$$

where $r_\chi = \deg(f_\chi)$ and f_χ is the homogeneous function corresponding to χ

- There is a particular character χ_T of $H_0^{\mathbb{C}}$ (coming from the complex structure of G/H_0) for which more is actually true Let $\tau := \tau_{\chi_T}$. Of course: $|\tau| \leq r_{\chi_T}(2g-2)$
- A Hodge bundle is called **maximal** if $|\tau| = r_{\chi_T}(2g-2)$

Cayley correspondence (Biquard–Collier–G–Toledo, 2018)

Let (E, φ) be a Hodge bundle with $|\tau| = r_{\chi_T}(2g-2)$. We have the following:
(1) $\operatorname{rank} \varphi(x) = r_{\chi_T}$ for every $x \in X$
(2) The Higgs field φ defines a reduction of $E \otimes \kappa$ to a $H_0'^{\mathbb{C}}$-bundle E' over X, where $H_0'^{\mathbb{C}}$ is the isotropy subgroup of an element in the open orbit of $\mathfrak{g}_1^{\mathbb{C}}$, and every $H_0'^{\mathbb{C}}$-bundle comes from a Hodge bundle with $|\tau| = r_{\chi_T}(2g-2)$
(3) Moreover, (E, φ) is (poly,semi)stable if and only if E' is (poly,semi)stable

- Suppose that a maximal Hodge bundle $(E_{H_0^{\mathbb{C}}}, \varphi_1)$ is a **minimum** of the Hitchin function, where recall φ_1 takes values in $\mathfrak{g}_1^{\mathbb{C}}$. Then consider a Higgs bundle of the form

$$(E, \varphi) = (E_{H_0^{\mathbb{C}}}, \varphi_1, \bigoplus_{k<0} \varphi_{2k+1})$$

with $\varphi_{2k+1} \in H^0(X, (E_{H_0^{\mathbb{C}}}(\mathfrak{g}_{2k+1}^{\mathbb{C}}) \otimes K)$

- There is a **Cayley correspondence** for these objects which sends (E, φ) to a pair (E', φ') where E' is an $H_0'^{\mathbb{C}}$-bundle and

$$\varphi' = \bigoplus_{k<0} \varphi'_{2k+1} := [\varphi_1, \bigoplus_{k<0} \varphi_{2k+1}]$$

- $(E_{H_0^{\mathbb{C}}}, \varphi_1)$ is a minimum is equivalent (**Bradlow–G–Gothen**, 2003) to having (H_0'-equivariant) isomorphisms

$$\mathrm{ad}(\varphi_1)\colon E(\mathfrak{g}_{2k}^{\mathbb{C}}) \otimes K \xrightarrow{\cong} E(\mathfrak{g}_{2k+1}^{\mathbb{C}}), \quad k < -1$$

- This implies that

$$\varphi'_{2k+1} \in H^0(X, (E_{H_0'^{\mathbb{C}}}(\mathfrak{g}_{2k}^{\mathbb{C}}) \otimes K^{1-k}) \quad \text{for} k < -1$$

On the other hand $\varphi'_1 \in H^0(X, (E_{H_0'^{\mathbb{C}}}(V) \otimes K^2$ where $V \oplus \mathfrak{h}_0'^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}}$

Claim (Bradlow–Collier–G–Gothen–Oliveira, work in progress

(1) If the dimension of the moduli space of objects $(E_{H_0^{\mathbb{C}}}, \varphi_1, \bigoplus_{k<0} \varphi_{2k+1})$ equals the dimension of $\mathcal{M}(X, G)$, then this moduli space is a union of higher Teichmüller components of $\mathcal{M}(X, G)$. The Cayley partner detects new topological invariants
(2) Moreover, the parabolic subgroup $P \subset G$ defining positivity on G in the sense of Guichard–Wienhard can be identified from the structure of the Cayley parter (the rigidity given by being a maximal minimum!). This identification will use the solution to the Hitchin equation, that is a reduction of the bundle E to an H-bundle.