LECTURES ON STRING TOPOLOGY OF CLASSIFYING SPACES

RICHARD HEPWORTH

These are notes for a series of three lectures to be given at the *Summer School on String Topology and Rational Homotopy Theory* in Hamburg, September 2nd-4th 2015. The subject is *string topology of classifying spaces*.

String topology began in 1999 with the paper of Chas and Sullivan [1], which found new algebraic structure on $H_*(LM)$, where M is a closed oriented manifold and LM is the space of free loops (the strings) in M. An excellent introduction can be found in Cohen and Voronov's notes [3], and the highpoint of the theory is perhaps Godin's paper [4] on higher string topology operations. String topology of classifying spaces was introduced by Chataur and Menichi [2], who studied the structure of $H_*(LBG)$ for compact Lie groups G. These three lectures are based on a paper of Anssi Lahtinen and myself [7], where we extend Chataur and Menichi's work. In the fourth lecture of this series, Lahtinen will explain his own work [8] that uses string topology to construct and detect homology classes in groups such as $F_n \rtimes \mathrm{Aut}(F_n)$.

Our aim in these three lectures is to answer the following question.

> **Question:** Let G be a finite group, let BG denote the classifying space of G, and let LBG denote the space of all maps from S^1 to BG. What is the structure of $H_*(BG)$ and $H_*(LBG)$?

Here, and throughout what follows, homology is taken with coefficients in a field $\mathbb{F}$. The answer that Chataur and Menichi gave to the question above is that $H_*(BG)$ and $H_*(LBG)$ are part of a *homological conformal field theory*, which is an algebraic structure governed by surfaces and their diffeomorphisms. However, Lahtinen and I found that only the most basic homotopical properties of surfaces are important, and so our answer to the question is that $H_*(BG)$ and $H_*(LBG)$ are part of what we call a *homological h-graph field theory*, which are similar to homological conformal field theories, but with surfaces and diffeomorphisms replaced with with more general (and frequently bizarre) spaces and their homotopy equivalences.

1. H-GRAPHS AND H-GRAPH COBORDISMS

This section will introduce h-graphs and h-graph cobordisms, which are our homotopical versions of 1-manifolds and surfaces, respectively. I hope that these sections are self-contained, but if you've never encountered field theoretical structures before then you might like to look at section 1.1 of Lurie's expository article [9].

Definition 1 (H-graph)**.** An *h-graph* is a space with the homotopy type of a finite CW-complex of dimension at most 1.

Example 2. Here are some basic examples of h-graphs.

Finite sets 1-manifolds Surfaces Wedges of S^1s

(The 1-manifolds must be compact, and the surfaces must have boundary in every component.)

Definition 3 (H-graph cobordism)**.** Let X and Y be h-graphs. An *h-graph cobordism*

$$S\colon X \twoheadrightarrow Y$$

consists of an h-graph S and a zig-zag of continuous maps

$$X \xrightarrow{i} S \xleftarrow{j} Y$$

satisfying the following conditions:

(i) $i \sqcup j\colon X \sqcup Y \to S$ is a closed cofibration.
(ii) $i(X)$ meets every path-component of S.
(iii) There is a homotopy cofibre square

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow j \\ B & \longrightarrow & S \end{array}$$

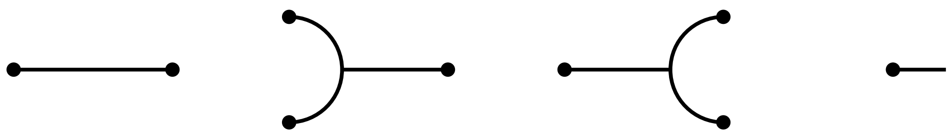
in which B is an h-graph and A has the homotopy type of a finite set.

Remark 4. Why do we impose conditions (i), (ii) and (iii) above? Condition (i) is necessary because later on we will want to consider homotopy automorphisms of S that respect X and Y, and without this restriction these automorphisms will behave in the wrong way. Condition (ii) is 'optional' in some sense, but without it we wouldn't be able to include string topology of classifying spaces as an example. And condition (iii) is the very heart of the definition: it is a homotopy-theoretical property of surfaces, and it is the *only* property of surfaces that we will need.

Example 5 (Graphs as h-graph cobordisms)**.** Let S be a finite graph, let X and Y be finite sets, and let $i\colon X \to S$ and $j\colon Y \to S$ be injections with disjoint images, such that $i(X)$ meets every component of S. Then S, i and j determine an h-graph cobordism

$$S\colon X \twoheadrightarrow Y.$$

Conditions (i), (ii) and (iii) are easily verified. Here are some specific examples. We let $\mathrm{pt} = \{p\}$ denote the space with a single point p.



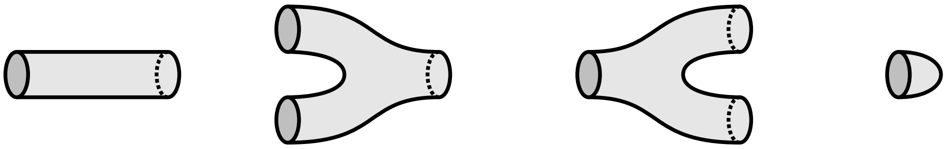
$i\colon \mathrm{pt} \twoheadrightarrow \mathrm{pt}$ $m\colon \mathrm{pt} \sqcup \mathrm{pt} \twoheadrightarrow \mathrm{pt}$ $w\colon \mathrm{pt} \twoheadrightarrow \mathrm{pt} \sqcup \mathrm{pt}$ $c\colon \mathrm{pt} \twoheadrightarrow \emptyset$

The pictures above just show the 'S' part of an h-graph cobordism $S\colon X \twoheadrightarrow Y$. The maps i and j, which we have not drawn, are the inclusion of the left and right 'ends' of S in the picture.

Example 6 (Surfaces as h-graph cobordisms)**.** Let X and Y be closed 1-manifolds and let S be a surface equipped with a diffeomorphism $i \sqcup j \colon X \sqcup Y \to \partial S$ between $X \sqcup Y$ and the boundary of S, such that $i(X)$ meets every component of S. Then S, i and j determine an h-graph cobordism

$$S \colon X \nrightarrow Y.$$

Here are some specific examples of h-graph cobordisms obtained in this way.



$I \colon S^1 \nrightarrow S^1$ $\quad$ $M \colon S^1 \sqcup S^1 \nrightarrow S^1$ $\quad$ $W \colon S^1 \nrightarrow S^1 \sqcup S^1$ $\quad$ $C \colon S^1 \nrightarrow \emptyset$

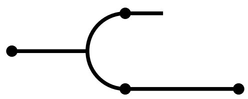
Conditions (i) and (ii) are easily verified. Condition (iii) is less trivial, and relies on the assumption that $i(X)$ meets every component.

Exercise 7. Verify condition (iii) for the h-graph cobordisms I, M, W and C in Example 6.

Exercise 8. Find h-graph cobordisms that are not of the form considered in Example 5 and 6. Include ones of the form $S^1 \nrightarrow \mathrm{pt}$, $\mathrm{pt} \nrightarrow S^1$ and $S^1 \to S^1$.

Definition 9 (Composites and disjoint unions)**.** Given h-graph cobordisms $S \colon X \nrightarrow Y$ and $T \colon Y \nrightarrow Z$, the *composite* $T \circ S \colon X \nrightarrow Z$ is defined by taking $T \circ S = T \cup_Y S$. And given $S_1 \colon X_1 \nrightarrow Y_1$ and $S_2 \colon X_2 \nrightarrow Y_2$, the *disjoint union* $S_1 \sqcup S_2 \colon X_1 \sqcup X_2 \nrightarrow Y_1 \sqcup Y_2$ is defined in the evident way.

Example 10. Taking the h-graph cobordisms c, i and w from Example 5, the composite $(c \sqcup i) \circ w$ is as follows.



Definition 11 (2-cells)**.** Let $S \colon X \nrightarrow Y$ and $S' \colon X' \nrightarrow Y'$ be h-graph cobordisms. A *2-cell* $\varphi \colon S \Rightarrow S'$ consists of three homotopy equivalences

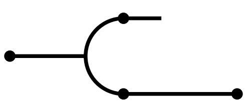
$$\varphi_X \colon X \xrightarrow{\simeq} X'$$
$$\varphi_Y \colon Y \xrightarrow{\simeq} Y'$$
$$\varphi_S \colon S \xrightarrow{\simeq} S'$$

compatible with the maps defining the h-graph cobordisms.

Example 12. There is a 2-cell

$$(c \sqcup i) \circ w \Rightarrow i.$$

For the domain and range are



$(c \sqcup i) \circ w$

i

and there is a homotopy equivalence between them that preserves the 'left' and 'right' copies of pt. If we regard w as a coproduct, then this 2-cell shows that we can regard c as a counit for w.

Exercise 13. Let $l\colon \mathrm{pt} \nrightarrow S^1$ and $d\colon S^1 \nrightarrow \mathrm{pt}$ be as follows.

$l\colon \mathrm{pt} \nrightarrow S^1$ $\qquad$ $d\colon S^1 \nrightarrow \mathrm{pt}$

Find 2-cells

$d\circ l \Rightarrow i \qquad (d\sqcup d)\circ W \Rightarrow w\circ d \qquad c\circ d \Rightarrow C \qquad W\circ l \Rightarrow (l\sqcup l)\circ w \qquad C\circ l \Rightarrow c$

and interpret them algebraically.

Exercise 14. Suppose that we remove condition (iii) from the definition of h-graph cobordism. Show that in this circumstance, the composite of two h-graph cobordisms need not be another h-graph cobordism.

2. Homotopy automorphisms

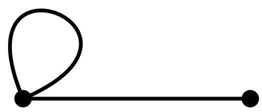
Definition 15 (Homotopy automorphisms)**.** Let $S\colon X \nrightarrow Y$ be an h-graph cobordism, determined by maps $i\colon X \to S$ and $j\colon Y \to S$. The *monoid of homotopy automorphisms of* S, denoted $\mathrm{hAut}(S)$, is defined to be the space of all homotopy equivalences

$$\alpha\colon S \longrightarrow S$$

that satisfy $\alpha\circ i = i$ and $\alpha\circ j = j$. It is a topological monoid under composition. Since $\mathrm{hAut}(S)$ is a monoid, so is $\pi_0(\mathrm{hAut}(S))$, and there is a homomorphism of monoids $\mathrm{hAut}(S) \to \pi_0(\mathrm{hAut}(S))$.

Proposition 16. *The morphism* $\mathrm{hAut}(S) \xrightarrow{\simeq} \pi_0(\mathrm{hAut}(S))$ *is a homotopy equivalence, and the monoid* $\pi_0(\mathrm{hAut}(S))$ *is a group.*

Example 17. Let $Q\colon \mathrm{pt} \nrightarrow \mathrm{pt}$ denote the following h-graph cobordism.



Then

$$\pi_0(\mathrm{hAut}(Q)) \cong \mathbb{Z} \ltimes \{\pm 1\}$$

where $\{\pm 1\}$ acts on $\mathbb{Z}$ by multiplication. The generator of $\{\pm 1\}$ corresponds to a homotopy automorphism that fixes the arc and flips the circle. The generator of $\mathbb{Z}$ corresponds to a homotopy automorphism that wraps the arc around the circle, and fixes the circle.

Exercise 18. Let $I\colon S^1 \to S^1$ denote the cylinder h-graph cobordism from Example 6. Show that

$$\pi_0(\mathrm{hAut}(I)) \cong \mathbb{Z}.$$

Hint: Choose a path in I that travels directly from the incoming S^1 to the outgoing S^1, and consider the effect of a homotopy automorphism on this path.

Exercise 19. Prove that there is an isomorphism $\pi_0(\mathrm{hAut}(Q)) \cong \mathbb{Z} \ltimes \{\pm 1\}$ as in Example 17.

Example 20. If $S\colon X \nrightarrow Y$ is obtained from a surface in as in Example 6, then a variant of the Dehn-Nielsen-Baer Theorem shows that the map $\mathrm{Diff}(S) \to \mathrm{hAut}(S)$ is a homotopy equivalence, and so $\pi_0(\mathrm{hAut}(S))$ is the mapping class group of S.

Example 21. If we modify the cobordism $Q\colon \mathrm{pt} \nrightarrow \mathrm{pt}$ of Example 17 so that it features a wedge of n circles, then $\pi_0(\mathrm{hAut}(Q))$ is the *holomorph* $F_n \ltimes \mathrm{Aut}(F_n)$ of the free group F_n on n letters.

Example 22. The free groups with boundary $A^s_{n,k}$ of Hatcher and Wahl [6] occur as $\pi_0(\mathrm{hAut}(S))$ for h-graph cobordisms $S\colon X \nrightarrow Y$ where S is a graph and X and Y are unions of circles and points.

When we can form composites and disjoint unions, there are maps

$$\mathrm{hAut}(T) \times \mathrm{hAut}(S) \longrightarrow \mathrm{hAut}(T \circ S)$$
$$\mathrm{hAut}(S_1) \times \mathrm{hAut}(S_2) \longrightarrow \mathrm{hAut}(S_1 \sqcup S_2)$$

And if there is a 2-cell $\varphi\colon S \Rightarrow S'$, then there is a zig-zag of homotopy equivalences of monoids

$$\mathrm{hAut}(S) \xleftarrow{\simeq} \mathcal{H} \xrightarrow{\simeq} \mathrm{hAut}(S').$$

3. Homological h-graph field theories

Definition 23. A *homological h-graph field theory* or *HHGFT* ϕ consists of a (strong) symmetric monoidal functor ϕ_* from the category of h-graphs and homotopy equivalences into the category of graded $\mathbb{F}$-vector spaces, and for each h-graph cobordism $S\colon X \nrightarrow Y$ a map

$$\phi(S)\colon H_*(B\mathrm{hAut}(S))) \otimes \phi_*(X) \longrightarrow \phi_*(Y).$$

These data are required to be compatible with *2-cells*, *composition*, *identity*, and *disjoint unions*.

Exercise 24. Formulate the compatibilities mentioned above. It may help to know that the composition axiom relates $\phi(S)$, $\phi(T)$ and $\phi(T \circ S)$, the disjoint union axiom relates $\phi(S_1)$, $\phi(S_2)$ and $\phi(S_1 \sqcup S_2)$, and the identity axiom gives some information about $\phi(I_X)$ where $I_X\colon X \nrightarrow X$ is the 'cylinder' h-graph cobordism.

Remark 25. The definition of what it means for ϕ_* to be symmetric monoidal can be read in chapters VII and XI of [10]. In particular ϕ_* gives us a graded vector space $\phi_*(X)$ for every h-graph X, and an isomorphism $\phi_\otimes\colon \phi_*(X) \otimes \phi_*(Y) \xrightarrow{\sim} \phi_*(X \sqcup Y)$ for any two h-graphs X and Y.

Remark 26. What does an HHGFT tell us? One point of view is that it is a rich algebraic structure on the spaces $\phi_*(X)$, which in our example will be the homology groups $H_*(BG^X)$. Another is that it gives us information about the homology of the spaces $B\mathrm{hAut}(S)$, which are interesting in their own right, and will be the subject of Lahtinen's final lecture in this series.

Definition 27 (Degree-0 operations)**.** Let ϕ be an HHGFT. Let $S\colon X \nrightarrow Y$ be an h-graph cobordism. The *degree-0 operation*

$$\phi_S\colon \phi_*(X) \longrightarrow \phi_*(Y)$$

is defined by $\phi_S(a) = \phi(S)(1 \otimes a)$ for $a \in \phi_*(X)$, where $1 \in H_0(B\mathrm{hAut}(S))$ denotes the canonical generator.

Proposition 28. *Degree-0 operations satisfy the following compatibilities.*

- *Given a 2-cell* $\varphi\colon S \Rightarrow S'$*, the square*

$$\begin{array}{ccc} \phi_*(X) & \xrightarrow{\phi_S} & \phi_*(Y) \\ \downarrow & & \downarrow \\ \phi_*(X') & \xrightarrow[\phi'_S]{} & \phi_*(Y') \end{array}$$

  commutes. (The unlabelled maps are obtained by applying the functor ϕ_* *to the homotopy equivalences* φ_X *and* φ_Y*.)*
- *Given h-graph cobordisms* $X \overset{S}{\nrightarrow} Y \overset{T}{\nrightarrow} Z$*, we have* $\phi_{T\circ S} = \phi_T \circ \phi_S$*.*
- *Given h-graph cobordisms* $S_1\colon X_1 \nrightarrow Y_1$ *and* $S_2\colon X_2 \nrightarrow Y_2$*, the following diagram commutes.*

$$\begin{array}{ccc} \phi_*(X_1)\otimes\phi_*(X_2) & \xrightarrow{\phi_{S_1}\otimes\phi_{S_2}} & \phi_*(Y_1)\otimes\phi_*(Y_2) \\ \phi_\otimes \downarrow & & \downarrow \phi_\otimes \\ \phi_*(X_1\sqcup X_2) & \xrightarrow[\phi_{S_1\sqcup S_2}]{} & \phi_*(Y_1\sqcup Y_2) \end{array}$$

  (The unlabelled maps come from monoidality of ϕ_**.)*
- *Given an h-graph X, and* $I_X\colon X \nrightarrow X$ *the 'cylinder' cobordism* $X \hookrightarrow X\times[0,1] \hookleftarrow X$*, then* ϕ_{I_X} *is the identity map.*

Example 29 ($\phi_*(\mathrm{pt})$ is a Frobenius algebra)**.** Let ϕ be an HHGFT. Then $\phi_*(\mathrm{pt})$ admits the structure of a non-unital commutative Frobenius algebra. The operations, obtained using the h-graph cobordisms m, w and i of Example 5, are as follows.

Product: $\phi_*(\mathrm{pt})\otimes\phi_*(\mathrm{pt}) \cong \phi_*(\mathrm{pt}\sqcup\mathrm{pt}) \xrightarrow{\phi_m} \phi_*(\mathrm{pt})$

Coproduct: $\phi_*(\mathrm{pt}) \xrightarrow{\phi_w} \phi_*(\mathrm{pt}\sqcup\mathrm{pt}) \cong \phi_*(\mathrm{pt})\otimes\phi_*(\mathrm{pt})$

Counit: $\phi_*(\mathrm{pt}) \xrightarrow{\phi_c} \phi_*(\emptyset) \cong \mathbb{F}$

The algebraic properties required to make $\phi_*(\mathrm{pt})$ into a Frobenius algebra now all follow from a combination of algebraic properties of the degree 0 operations and properties of the cobordisms m, w and c.

Exercise 30. Prove in detail that the product on $\phi_*(\mathrm{pt})$ is associative. *Hint:* The first step is to use the compatibilities, together with the rule $\phi_\otimes\circ(\phi_\otimes\otimes\mathrm{Id}) = \phi_\otimes\circ(\mathrm{Id}\otimes\phi_\otimes)$, to reduce this to the claim that $\phi_{m\circ(m\sqcup i)} = \phi_{m\circ(i\sqcup m)}$. The second step is to prove this by constructing a 2-cell $m\circ(m\sqcup i) \Rightarrow= m\circ(i\sqcup m)$.

Definition 31 (Higher operations)**.** If we have an h-graph cobordism $S\colon X \nrightarrow Y$ and an element $\sigma \in H_i(B\mathrm{hAut}(S))$ with $i > 0$, then the associated *higher operation*

$$\phi_*(X) \longrightarrow \phi_{*+i}(Y)$$

is defined by $a \mapsto \phi(S)(\sigma\otimes a)$.

Example 32. The *BV operator* $\Delta\colon \phi_*(S^1) \to \phi_{*+1}(S^1)$ is defined to be the higher associated to the generator $\sigma \in H_1(B\mathrm{hAut}(S^1\times[0,1])) \cong \mathbb{Z}$. It satisfies $\Delta\circ\Delta = 0$.

4. TRANSFER MAPS

The main result, which will appear in the next section, is the existence of an HHGFT whose value on S^1 is $H_*(LBG)$. In order to construct this HHGFT we need to use a theory of 'umkehr' or 'wrong-way' maps in homology. Since our input is a finite group G, it turns out that we will only need the classical transfer map. A simple reference for this is Section 3.G of [5].

Definition 33 (The transfer)**.** Let $\pi\colon E \to B$ be a finite sheeted covering space. Then the *transfer map*

$$\pi^*\colon H_*(B) \longrightarrow H_*(E)$$

is the map induced by the chain map $C_*(B) \to C_*(E)$ that sends a singular simplex $\sigma\colon \Delta^k \to B$ to the sum of its distinct lifts $\tilde{\sigma}\colon \Delta^k \to E$.

In the definition above we do not insist that the cardinality of the fibres is constant, though it is necessarily locally constant. Nevertheless, if $\pi\colon E \to B$ is an n-sheeted covering, so that every fibre has cardinality n, then every singular simplex in B has precisely n distinct lifts, and consequently the composite

$$H_*(B) \xrightarrow{\pi^*} H_*(E) \xrightarrow{\pi_*} H_*(B)$$

is multiplication by n.

Example 34. Let $\pi\colon S^1 \to S^1$ denote the 2-sheeted covering map. Then $\pi^*\colon H_0(S^1) \to H_0(S^1)$ is multiplication by 2, while $\pi^*\colon H_1(S^1) \to H_1(S^1)$ is the identity map. This can be verified directly from the definition. It also follows by computing π_* and using fact that $\pi_* \circ \pi^*$ is multiplication by 2.)

Exercise 35. Let p be a prime, and let $\pi\colon E \to B$ be an n-sheeted covering, where n is coprime to p. Show that the induced map $\pi_*\colon H_*(E;\mathbb{Z}/p\mathbb{Z}) \to H_*(B;\mathbb{Z}/p\mathbb{Z})$ is injective.

We now collect some standard results that will be important in what follows. They are all routine.

Proposition 36 (Formal properties of the transfer)**.** *The transfer map satisfies the following properties.*

Naturality: *Suppose given the commutative diagram on the left, where π, π' are finite-sheeted covering spaces and F is an isomorphism on fibres.*

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow[f]{} & B' \end{array} \qquad\qquad \begin{array}{ccc} H_*(E) & \xrightarrow{F_*} & H_*(E') \\ {\scriptstyle\pi^*}\uparrow & & \uparrow{\scriptstyle(\pi')^*} \\ H_*(B) & \xrightarrow[f_*]{} & H_*(B') \end{array}$$

Products: *Suppose given finite-sheeted covering spaces $\pi_1\colon E_1 \to B_1$ and $\pi_2\colon E_2 \to B_2$. Then $\pi_1 \times \pi_2\colon E_1 \times E_2 \to B_1 \times B_2$ is also a finite-sheeted covering, and* *the following diagram commutes.*

$$\begin{array}{ccc} H_*(E_1)\otimes H_*(E_2) & \xrightarrow{\times} & H_*(E_1\times E_2) \\ {\scriptstyle \pi_1^*\otimes\pi_2^*}\big\uparrow & & \big\uparrow{\scriptstyle (\pi_1\times\pi_2)^*} \\ H_*(B_1)\otimes H_*(B_2) & \xrightarrow[\times]{} & H_*(B_1\times B_2) \end{array}$$

Composites: *Suppose given finite-sheeted coverings $\rho\colon R\to E$ and $\pi\colon E\to B$. Then $\pi\circ\rho\colon R\to B$ is also a finite-sheeted covering, and $(\pi\circ\rho)^*=\rho^*\circ\pi^*$.*

Identity: *If B is any space, then the identity map $\mathrm{Id}\colon B\to B$ is a 1-sheeted covering, and $\mathrm{Id}^*=\mathrm{Id}$.*

5. STRING TOPOLOGY OF BG

Theorem 37 (Hepworth-Lahtinen)**.** *Let G be a finite group. Then there is an HHGFT ϕ for which $\phi_*(X)=H_*(BG^X)$ for any h-graph X, where BG^X denotes the space of all maps from X to BG. In particular, $\phi_*(\mathrm{pt})=H_*(BG)$ and $\phi_*(S^1)=H_*(LBG)$.*

We will not prove the theorem. Instead, we will just sketch how to construct the *string topology operation*

$$\phi(S)\colon H_*(B\mathrm{hAut}(S))\otimes H_*(BG^X)\longrightarrow H_*(BG^Y).$$

The h-graph cobordism $S\colon X\nrightarrow Y$ is determined by maps

$$X\xrightarrow{i}S\xleftarrow{j}Y$$

which induce restriction maps

$$BG^X\longleftarrow BG^S\longrightarrow BG^Y.$$

There is a parameterised version

$$B\mathrm{hAut}(S)\times BG^X\xleftarrow{\alpha}B\mathrm{hAut}(S)\times_{\mathrm{twisted}}BG^S\xrightarrow{\beta}B\mathrm{hAut}(S)\times BG^Y$$

(the symbol in the middle is just a name) consisting of spaces and maps over $B\mathrm{hAut}(S)$ whose fibres are in some sense copies of the previous sequence, twisted according to the action of $\mathrm{hAut}(S)$. Now α is not a finite-sheeted covering. However, we will see in the next section how to 'replace' it with a finite-sheeted covering space. This is enough for us to construct a *transfer map*

$$\alpha^*\colon H_*(B\mathrm{hAut}(S)\times BG^X)\longrightarrow H_*(B\mathrm{hAut}(S)\times_{\mathrm{twisted}}BG^S).$$

Then $\phi(S)$ is defined to be the composite

$$\begin{aligned} H_*(B\mathrm{hAut}(S))\otimes H_*(BG^X) &\xrightarrow{\times} H_*(B\mathrm{hAut}(S)\times BG^X)\\ &\xrightarrow{\beta_*\circ\alpha^*} H_*(B\mathrm{hAut}(S)\times BG^Y)\\ &\xrightarrow{(\pi_2)_*} H_*(BG^Y). \end{aligned}$$

The rest of these lectures will tell you how to really compute these things.

6. HOMOTOPY QUOTIENTS

Definition 38 (Pairs)**.** We will consider pairs (X,G) consisting of a (discrete) group G and a G-set X, i.e. a set X with an action of G. A *map of pairs* $(X,G) \to (Y,H)$ consists of $f\colon X \to Y$ and $\varphi\colon G \to H$ satisfying $f(g\cdot x) = \varphi(g)\cdot f(x)$ for $x \in X$ and $g \in G$.

Definition 39 (Homotopy quotient)**.** Let G be a (discrete) group and let X be a G-set, or in other words a set with G-action. The *homotopy quotient* $X/\!/G$ is defined by

$$X/\!/G = (EG \times X)/G$$

where G acts diagonally on $EG \times X$. The assignment $(X,G) \mapsto X/\!/G$ is functorial.

Example 40. $\mathrm{pt}/\!/G = BG$

Example 41. Let $H \subset G$ and consider the G-set G/H. There is a map of pairs $(\mathrm{pt},H) \to (G/H,G)$ determined the inclusion $H \hookrightarrow G$ and the map $\mathrm{pt} \to G/H$ that sends the point to eH. The induced map

$$BH = \mathrm{pt}/\!/H \xrightarrow{\simeq} (G/H)/\!/G$$

is a homotopy equivalence.

Example 42 (Orbits and stabilizers)**.** Let G be a group and let X be an arbitrary G-set. Given $x \in X$ we write Gx for the orbit and G_x for the stabiliser. Then

$$X = \bigsqcup Gx \cong \bigsqcup G/G_x,$$

where x ranges over a set of orbit representatives. Consequently the homotopy quotient

$$X/\!/G = \bigsqcup Gx/\!/G \cong \bigsqcup (G/G_x)/\!/G \simeq \bigsqcup BG_x,$$

is simply the disjoint union of the classifying spaces of the stabilisers of the orbit representatives.

7. HOW TO UNDERSTAND BG^X

Definition 43 (Basepoints)**.** A set of *basepoints* for an h-graph X is a finite subset $P \subset X$ that contains at least one point in each path component and for which $P \hookrightarrow X$ is a cofibration. A *map of h-graphs with basepoints* $f\colon (X,P) \to (Y,Q)$ is a map $f\colon X \to Y$ sending P into Q.

Definition 44 (Fundamental groupoid etc.)**.** Let X be an h-graph with basepoints $P \subset X$.

- $\Pi_1(X,P)$ denotes the fundamental groupoid of X with basepoints in P. Its objects are the points of P and its morphisms are the path-homotopy classes of paths in X with basepoints in P.
- G^P is the set of functions $g\colon P \to G$. It is a group under pointwise multiplication. (It is the product of $\#P$ copies of G.)
- $G^{\Pi_1(X,P)}$ is the set of functions $f\colon \mathrm{Mor}(\Pi_1(X,P)) \to G$ satisfying $f(\delta\cdot\gamma) = f(\delta)f(\gamma)$ whenever δ,γ are composable morphisms in $\Pi_1(X,P)$.
- G^P acts on $G^{\Pi_1(X,P)}$ as follows. Given $g \in G^P$ and $f \in G^{\Pi_1(X,P)}$, we define $g\cdot f$ by the rule $(g\cdot f)(\gamma) = g(q)f(\gamma)g(p)^{-1}$ for $\gamma\colon p \to q$ in $\Pi_1(X,P)$.

- A map of h-graphs with basepoints $f\colon (X,P) \to (Y,Q)$ induces a map of pairs $(G^Q, G^{\Pi_1(Y,Q)}) \to (G^P, G^{\Pi_1(X,P)})$.

Theorem 45 (Model of the mapping space)**.** *Let X be an h-graph with basepoints P. Then there is a zig-zag of homotopy equivalences*

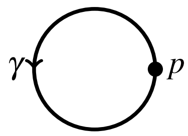
$$BG^X \longleftrightarrow G^P /\!/ G^{\Pi_1(X,P)}.$$

Given $f\colon (X,P) \to (Y,Q)$, the "square"

$$\begin{array}{ccc} BG^Y & \longrightarrow & BG^X \\ \simeq \updownarrow & & \updownarrow \simeq \\ G^Q /\!/ G^{\Pi_1(Y,Q)} & \longrightarrow & G^P /\!/ G^{\Pi_1(X,P)} \end{array}$$

commutes.

Example 46. Let us study $LBG = \mathrm{Map}(S^1, BG)$ using the theorem above. Let $X = S^1$, let $P = \{p\}$ where p is any point of S^1, and let γ denote the path homotopy class of a loop that starts and ends at p and travels once around S^1.



Applying Theorem 45 gives us a homotopy equivalence

$$LBG \simeq G^{\Pi_1(S^1,P)} /\!/ G^P.$$

If we let G^{ad} denote G equipped with the conjugation action, then there are isomorphisms

$$G^P \xrightarrow{\cong} G, \ \ g \mapsto g(p) \qquad\qquad G^{\Pi_1(S^1,P)} \xrightarrow{\cong} G^{\mathrm{ad}}, \ \ f \mapsto f(\gamma)$$

that form a map of pairs. So in fact the homotopy equivalence has the form

$$LBG \simeq G^{\mathrm{ad}} /\!/ G.$$

Next we study the right hand side in more detail. The orbits of G on G^{ad} are precisely the conjugacy classes of G, and the stabilizer of $h \in G^{\mathrm{ad}}$ is the centralizer $C(h)$. So $G^{\mathrm{ad}} \cong \bigsqcup G/C(h)$, and consequently

$$G^{\mathrm{ad}} /\!/ G \cong \bigsqcup BC(h)$$

where h ranges over a set of representatives for the conjugacy classes of G.

Exercise 47. Describe $BG^{S^1 \vee S^1}$ in terms of the *double conjugacy classes* and *double centralizers* of G, which are the orbits and stabilizers of the action of G on $G \times G$ defined by $g \cdot (h_1, h_2) = (gh_1g^{-1}, gh_2g^{-1})$.

8. HOW TO COMPUTE DEGREE-0 OPERATIONS

Now we will explain how to compute the degree-0 operation

$$\phi_S\colon H_*(BG^X) \longrightarrow H_*(BG^Y)$$

associated to an h-graph cobordism $S\colon X \nrightarrow Y$. Choosing basepoints $P \subset X$ and $Q \subset Y$, and using Theorem 45 to replace the domain and range results in an operation

$$\overline{\phi}_S\colon H_*(G^{\Pi_1(X,P)}//G^P) \longrightarrow H_*(G^{\Pi_1(Y,Q)}//G^Q),$$

and this is what we will describe.

Proposition 48. *Let $S\colon X \nrightarrow Y$ be an h-graph cobordism. Then*

$$\overline{\phi}_S = \nu_* \circ (\mu_*)^{-1} \circ \lambda^*$$

Here α, β and γ are the maps of homotopy quotients

$$\begin{array}{ccc} G^{\Pi_1(X,P)}//G^P & \xleftarrow{\lambda} & G^{\Pi_1(S,P)}//G^P \\ & & \mu \uparrow \simeq \\ & & G^{\Pi_1(S,P\sqcup Q)}//G^{P\sqcup Q} \xrightarrow{\nu} G^{\Pi_1(Y,Q)}//G^Q \end{array}$$

induced by the following maps of h-graphs with basepoints.

$$\begin{array}{ccc} (X,P) & \xrightarrow{l} & (S,P) \\ & & m \downarrow \\ & & (S,P\sqcup Q) \xleftarrow{n} (Y,Q) \end{array}$$

Here λ is a finite-sheeted covering because $G^{\Pi_1(X,P)}$ and $G^{\Pi_1(S,P)}$ are finite sets being acted on by the same finite group G^P. This allows us to form λ^*. And μ is a homotopy equivalence because its domain and range are both models for BG^S. This allows us to form $(\mu_*)-1$.

Remark 49. In practice it seems that the hardest challenge in computing one of these operations is to invert μ_*. We do this (when possible) by first decomposing the domain and range of μ as a disjoint union according to the orbits and stabilizers of the actions of G^P on $G^{\Pi_1(S,P)}$ and of $G^{P\sqcup Q}$ on $G^{\Pi_1(S,P\sqcup Q)}$.

Example 50 (The Frobenius algebra $H_*(BG)$)**.** The HHGFT ϕ endows $\phi_*(\mathrm{pt}) = H_*(BG)$ with the structure of a commutative Frobenius algebra with counit. The product, coproduct and counit for this Frobenius algebra structure are given as follows.

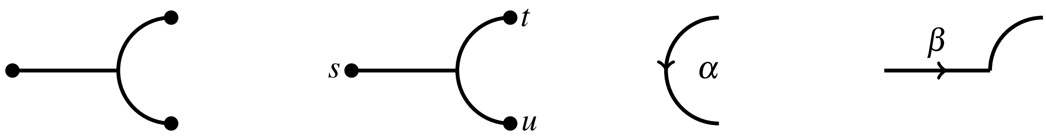
$$H_*(BG) \otimes H_*(BG) \xrightarrow[\cong]{\times} H_*(BG \times BG) \xrightarrow{\Delta^*} H_*(BG) \tag{1}$$

$$H_*(BG) \xrightarrow{\Delta_*} H_*(BG \times BG) \xrightarrow[\cong]{\times^{-1}} H_*(BG) \otimes H_*(BG) \tag{2}$$

$$H_*(BG) \xrightarrow{c_*} H_*(\mathrm{pt}) \cong \mathbb{F} \tag{3}$$

Here $\Delta\colon BG \to BG \times BG$ is the diagonal and $c\colon BG \to \mathrm{pt}$ is the constant map. The former 'is' a covering space if we regard BG as $(EG \times EG)/G$ and $BG \times BG$ as $(EG \times EG)/(G \times G)$.

Let us derive the formula for the coproduct. Comparing with Example 29, we must show that $\overline{\phi}_w = \Delta_*$, where $w\colon \mathrm{pt} \nrightarrow \mathrm{pt}\sqcup\mathrm{pt}$ is the h-graph cobordism on the left.



Choose basepoints $P=\{s\}$ for pt, $Q=\{t,u\}$ for $\mathrm{pt}\sqcup\mathrm{pt}$, and let α and β be the path homotopy classes of the paths depicted. Then $\bar{\phi}_w$ is determined by the following zig-zag.

$$\begin{array}{ccccc} G^{\Pi_1(\mathrm{pt},P)}/\!/G^P & \xleftarrow{\lambda} & G^{\Pi_1(w,P)}/\!/G^P & & \\ & & \mu\uparrow\simeq & & \\ & & G^{\Pi_1(w,P\sqcup Q)}/\!/G^{P\sqcup Q} & \xrightarrow{\nu} & G^{\Pi_1(\mathrm{pt}\sqcup\mathrm{pt},Q)}/\!/G^Q \end{array}$$

Let us simplify the spaces appearing here. We will spell things out for $G^{\Pi_1(w,P\sqcup Q)}/\!/G^{P\sqcup Q}$ and leave the rest as an exercise. First, we write $G^{P\sqcup Q}$ as $G^s\times G^t\times G^u$, which is just a product of copies of G labelled by the corresponding basepoints. Next, observe that in $\Pi_1(w,P\sqcup Q)$ the morphisms are the three identity morphisms, plus α, β, $\alpha\circ\beta$ and their inverses. It follows that a function $f\colon \mathrm{Mor}(\Pi_1(w,P\sqcup Q))\to G$ satisfying $f(\sigma\circ\tau)=f(\sigma)\circ f(\tau)$ is freely determined by $f(\alpha)$ and $f(\beta)$. In other words, we have an isomorphism $G^{\Pi_1(w,P\sqcup Q)}\to G^\alpha\times G^\beta$ given by $f\mapsto (f(\alpha),f(\beta))$. The action of $G^s\times G^t\times G^u$ on $G^\alpha\times G^\beta$ is determined by the endpoints of α and β, in the sense that

$$(g_s,g_t,g_u)\cdot(h_\alpha,h_\beta) = (g_u g_\alpha g_t^{-1}, g_t h_\beta g_s^{-1}).$$

Carrying out a similar analysis for the remaining terms, we can rewrite the zig-zag above as

$$\begin{array}{ccccc} \mathrm{pt}/\!/G^s & \xleftarrow{\lambda} & \mathrm{pt}/\!/G^s & & \\ & & \mu\uparrow\simeq & & \\ & & (G^\alpha\times G^\beta)/\!/(G^s\times G^t\times G^u) & \xrightarrow{\nu} & \mathrm{pt}/\!/(G^t\times G^u) \end{array}$$

Now the action of $G^s\times G^t\times G^u$ on $G^\alpha\times G^\beta$ is transitive, and the stabilizer of the point (e,e) is just the diagonal subgroup, which we denote by G^{stu}. This means that we can replace $(G^\alpha\times G^\beta)/\!/(G^s\times G^t\times G^u)$ with the homotopy equivalent $\mathrm{pt}/\!/G^{stu}$, and the zig-zag now has the form

$$\begin{array}{ccccc} \mathrm{pt}/\!/G^s & \xleftarrow{\lambda} & \mathrm{pt}/\!/G^s & & \\ & & \mu\uparrow & & \\ & & \mathrm{pt}/\!/G^{stu} & \xrightarrow{\nu} & \mathrm{pt}/\!/(G^t\times G^u). \end{array}$$

Identifying the all the groups with G itself, we find that λ and μ are just the identity map, while $\nu=\Delta$ is the diagonal. Consequently

$$\bar{\phi}_w = \nu_*\circ(\mu_*)^{-1}\circ\lambda^* = \Delta_*\circ(\mathrm{Id}_*)^{-1}\circ\mathrm{Id}^* = \Delta_*$$

as required.

Exercise 51. Use the general description of the degree 0 operations to derive the formulas (1) and (3) for the product and counit on $H_*(BG) \cong H_*(G)$.

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