Deformation Theory

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CHAPTER 1

Analytic algebras

1.1. On the dimension of an analytic algebra

Let $A = R/I$ be the quotient of a local ring $(R, \mathfrak{m}_R, K)$ modulo an ideal $I \subseteq \mathfrak{m}_R$. Denoting $\mu(I) = \dim_K I/\mathfrak{m}_R I$ the minimal number of generators of I, the (Krull) dimension of A is trivially bounded from below through

$$\dim A \geq \dim R - \mu(I)\,.$$

Much better estimates are possible, for example

$$\dim A \geq \dim R - s(I) \geq \dim R - \mu(I) + \mu(\overline{\mathfrak{m}I} \cap I) \tag{1}$$

where $s(I)$ is the *analytic spread* of the ideal I and where $\overline{\mathfrak{m}I}$ denotes the *integral closure* of $\mathfrak{m}I$ in R. We review these notions and prove the inequalities below. Both estimates are geometrically motivated. The difference $\dim R - \dim A$ is the codimension of the zero set $V(I)$ of I as a closed subspace in the space underlying R, the analytic spread is the dimension of the special fibre of the affine blow-up of I in R, and the lower bound $s(I)$ amounts to semicontinuity of the fibre dimension of a morphism between spaces. The geometric intuition behind the lower estimate involving $\overline{\mathfrak{m}I}$ is a "curve test": If we know all germs of curves contained in the germ underlying A, then we know its dimension. Unfortunately, both blow-up and integral closure of an ideal are in general difficult to determine if little is known about the ideal.

If A is an *analytic algebra* over K and K is of characteristic zero, a simple compromise is possible: Choosing for R a smooth analytic K-algebra and denoting $\Omega^1_{A/K}$ the module of universally finite differential forms, one has

$$\mu(I) \geq \dim_K \operatorname{Ext}^1_A(\Omega^1_{A/K}, K) \geq \mu(I) - \mu(\overline{\mathfrak{m}I} \cap I)\,. \tag{2}$$

That the codimension of an analytic algebra can be bounded from below in terms of $\dim_K \operatorname{Ext}^1_A(\Omega^1_{A/K}, K)$ is a result due to Scheja-Storch [**SSto**, (3.5)] who interpret the latter dimension as "differential defect" of A and call it the *differential rank* or *d-rank* of I in R.

Before proving the results in full, let us remark rightaway why the left inequality in (2) holds without restrictions on K. Consider the Jacobi map $\mathrm{jac}\colon I/I^2 \to \Omega^1_{R/K} \otimes_R A$ and let $J = \mathrm{Im}(\mathrm{jac})$ be its image. Applying $\operatorname{Hom}_A(-, K)$ to the exact sequence

$$0 \to J \to \Omega^1_{R/K} \otimes_R A \to \Omega^1_{A/K} \to 0$$

yields a surjection $\operatorname{Hom}_A(J, K) \to \operatorname{Ext}^1_A(\Omega^1_{A/K}, K)$ as $\Omega^1_{R/K} \otimes_R A$ is a free A-module. By definition, $\operatorname{Hom}_A(J, K) \subseteq \operatorname{Hom}_A(I/I^2, K) \cong \operatorname{Hom}_K(I/\mathfrak{m}I, K)$ and so $\mu(I) \geq \dim_K \operatorname{Ext}^1_A(\Omega^1_{A/K}, K)$ as stated.

To formulate the other claims succinctly, recall that the *embedding dimension* of a local ring $(A,\mathfrak{m},K)$ is defined as $\operatorname{emdim} A = \dim_K \mathfrak{m}/\mathfrak{m}^2$ and that the *analytic spread* of I is the (Krull) dimension $s(I) = \dim gr_I(R)\otimes_{R/I} K$ where $gr_I(R)$ is the graded R/I–algebra associated to the I-adic filtration on R.

Theorem 1.1.1. *If $A = R/I$ is a quotient of a regular local ring $(R,\mathfrak{m},K)$ modulo an ideal $I \subseteq \mathfrak{m}$, then*

$$\dim A \geq \operatorname{emdim} A - s(I) \geq \operatorname{emdim} A - \mu(I) + \mu(\overline{\mathfrak{m}I} \cap I)\,.$$

If A is an analytic K–algebra over a field K of characteristic zero, then

(1) $\operatorname{emdim} A = \dim_K \operatorname{Hom}_A(\Omega^1_{A/K}, K)$

(2) $\dim A \geq \operatorname{emdim} A - \dim_K \operatorname{Ext}^1_A(\Omega^1_{A/K}, K)\,.$

Note that 1.1.1(2) can fail in positive characteristic. For example, if K is a field of characteristic $p>0$, then $A = K_{x_1,\dots,x_n}/(x_1^p,\dots,x_n^p)$ is of Krull dimension zero and of embedding dimension n, but $\Omega^1_{A/K}$ is A–free so that $\operatorname{Ext}^1_A(\Omega^1_{A/K},K)=0$. According to [**SSto**, (3.2)], the estimate still holds if K is perfect and A is generically reduced.

Before proving the theorem we recall first the definition of integral dependence, see [**1**].

Definition 1.1.2. Let R be a ring and $I \subseteq R$ an ideal. An element $x \in R$ is *integral* over I if there is an equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

with $a_\nu \in I^\nu$.

For instance, every element of I is integral over I. The set $\overline{I} \subseteq R$ of all elements from R that are integral over I is an ideal, the *integral closure* of I in R.

An important classical criterion for integral dependence is in terms of valuations. It formulates algebraically the geometric intuition that integral dependence can be tested along curves.

Theorem 1.1.3. *Let R be a normal Noetherian ring and $I \subseteq R$ an ideal. An element $f \in R$ is integral over I iff for every ring homomorphism $\varphi\colon R \to V$ into a discrete valuation ring V with valuation $\mathbf{v}$ the inequality*

$$\mathbf{v}(\varphi(f)) \geq \min\{\mathbf{v}(\varphi(g)) : g \in I\}$$

holds, that is $\varphi(f) \in \varphi(I)V$.

For a proof we refer the reader to [**2**, p.353, Theorem 3]. □

Remark 1.1.4. In the complex analytic case, this criterion can be reduced to the following geometric one: If $(X,0)$ is the germ of a reduced complex space and $R = \mathcal{O}_{X,0}$, then f is in the integral closure of $I \subseteq m_R$ iff for every complex arc $\varphi\colon (\mathbb{C},0) \to (X,0)$ it is true that $\varphi^*(f) \in \varphi^*(I)\mathcal{O}_{\mathbb{C},0}$ where $\varphi^*\colon \mathcal{O}_{X,0} \to \mathcal{O}_{\mathbb{C},0}$ is the associated homomorphism of analytic $\mathbb{C}$–algebras. In other words, along every arc the vanishing order of f at 0 is at least as large as the vanishing order of some function from I.

We split the proof of the theorem 1.1.1 into a sequence of lemmata and propositions. The first part is a consequence of the following.

PROPOSITION 1.1.5. *Let* $(R, \mathfrak{m}, K)$ *be a local ring and* $J \subseteq I \subseteq \mathfrak{m}$ *ideals such that* J *is integral over* $\mathfrak{m}I$ *and* $I/(J + \mathfrak{m}I)$ *is a* K*–vector space of dimension* k*. Then* $\dim R/I \geq \dim R - k$*. In particular,*

$$\dim R/I \geq \dim R - \dim_K I/(\overline{\mathfrak{m}I} \cap I) = \dim R - \mu(I) + \mu(\overline{\mathfrak{m}I} \cap I)\,.$$

PROOF. Replacing J by $J + \mathfrak{m}I$ we may assume that $J \supseteq \mathfrak{m}I$. Choose elements $x_1, \ldots, x_k \in I$ that form a basis of the K-vector space I/J and consider the natural ring homomorphism

$$K[X_1, \ldots, X_k] \longrightarrow \bigoplus_{\nu=0}^{\infty} (I^\nu/\mathfrak{m}I^\nu)\, T^\nu = R[IT]/\mathfrak{m}R[IT]$$

given by $X_i \mapsto \overline{x_i}T \in (I/\mathfrak{m}I)T$.

In a first step we prove that this map is finite. In fact, the elements $\bar{f}T\,, f \in J$, generate the ring $R[IT]/\mathfrak{m}R[IT]$ as an algebra over $K[X_1, \ldots, X_k]$, and if $f^n + a_1 f^{n-1} + \cdots + a_n = 0$ is an equation of integral dependance for such an $f \in J$ over $\mathfrak{m}I$, the coefficients satisfy $a_\nu \in (\mathfrak{m}I)^\nu$, whence $(\bar{f}T)^n = 0$ and finiteness follows. This implies

$$\dim R[IT]/\mathfrak{m}R[IT] \leq k$$

and it suffices thus to show that

$$\dim R - \dim R/I \leq \dim R[IT]/\mathfrak{m}R[IT]\,.$$

But $R[IT]/\mathfrak{m}R[IT]$ appears as the special fibre of

$$R/I \longrightarrow gr_I(R) = \bigoplus_{\nu=0}^{\infty} I^\nu/I^{\nu+1}$$

and so, by [**Mat**, Thms.15.1, 15.7],

$$\dim R[IT]/\mathfrak{m}R[IT] \geq \dim gr_I(R) - \dim R/I = \dim R - \dim R/I\,.$$

□

By definition, the dimension of the ring $R[IT]/\mathfrak{m}R[IT] \cong gr_I \otimes_{R/I} K$ is the analytic spread $s(I)$ of I. The above proof thus shows

COROLLARY 1.1.6. *With notation as in the preceding proposition,*

$$ht(I) \leq s(I) \leq k \leq \mu(I) - \mu(\overline{\mathfrak{m}I} \cap I)$$

or, equivalently,

$$\dim R/I \geq \dim R - s(I) \geq \dim R - \mu(I) + \mu(\overline{\mathfrak{m}I} \cap I)\,.$$

□

Also the mentioned geometric meaning of $s(I)$ is apparent from the proof of the proposition: It is the dimension of the special fibre in the affine blow-up of the space underlying R along $V(I)$, the closed subspace underlying A. The lower bound in terms of the analytic spread then just says that the dimension of the special fibre is at least as large as the codimension of A in R.

Now we turn to the easier differential estimate and prove the second part of the theorem where we may assume that $K = \mathbb{C}$. We begin with the following local version of Sard's theorem that expresses the generic smoothness of a function.

Lemma 1.1.7. *Let* $f \in \mathbb{C}\{x_1, \dots, x_n\}$ *be a function and denote* $\operatorname{jac} f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ *the Jacobi ideal of* f. *If* $f(0) = 0$, *then* $f \in \sqrt{\operatorname{jac} f}$.

Proof. Let C be the reduced critical set of f, that is the set of zeros of $\sqrt{\operatorname{jac} f}$. Its analytic algebra at 0 is accordingly $\mathcal{O}_{C,0} = \mathcal{O}_{\mathbb{C}^n,0}/\sqrt{\operatorname{jac} f}$. If f does not vanish on $(C, 0)$, then one can find a curve $\varphi\colon (\mathbb{C}, 0) \to (C, 0)$ such that $f\varphi$ is not identically zero. Algebraically speaking, if $f \neq 0$ in the reduced local ring $\mathcal{O}_{C,0}$, then C is of dimension at least one and there exists a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_{C,0}$ that is of codimension one and does not contain f. The normalization $\mathcal{O}_{C,0}/\mathfrak{p} \subseteq \mathbb{C}\{t\}$ defines then such a curve. As $\varphi^* f \in \mathbb{C}\{t\}$ is not the zero function, but satisfies $\varphi^* f(0) = 0$, its derivative $\partial\varphi^* f/\partial t$ with respect to t does not vanish identically. Applying the chain rule to $\varphi^* f \in \mathbb{C}\{t\}$ yields now a contradiction:

$$\frac{\partial \varphi^* f}{\partial t} = \sum_{i=1}^{n} \varphi\left(\frac{\partial f}{\partial x_i}\right) \frac{d\varphi(x_i)}{dt} = 0 \in \mathbb{C}\{t\}$$

as by choice of φ one has $\varphi(\partial f/\partial x_i) = 0$ for each i. □

As an application we get the following result, see [**BSk**, Cor.], [**Tei**, Exerc.3, p.591], that is a generalization of Euler's identity for homogeneous polynomials.

Proposition 1.1.8. *A function* $f \in \mathbb{C}\{x_1, \dots, x_n\}$ *with* $f(0) = 0$ *is integral over the ideal* $I := \left(x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n}\right)$. *In particular,* f *is integral over* $\mathfrak{m} \operatorname{jac}(f)$.

Proof. We first show that f belongs to the radical of I. Denoting $S \subseteq \{1, \dots, n\}$ any subset, the ideal $j_S := (\frac{\partial f}{\partial x_j}; j \notin S) + (x_i; i \in S)$ contains I and $\sqrt{I} = \bigcap_S \sqrt{j_S}$. With f_S the image of f in $\mathbb{C}\{x_1, \dots, x_n\}/\left(x_i; i \in S\right)$, one has

$$j_S \equiv \left(\frac{\partial f}{\partial x_j}; j \notin S\right) \equiv \left(\frac{\partial f_S}{\partial x_j}; j \notin S\right) = (\operatorname{jac} f_S) \bmod \left(x_i; i \in S\right).$$

As $f_S \in \sqrt{\operatorname{jac} f_S}$ for every subset S by the preceding lemma, it follows that

$$f \in (f_S) + (x_i; i \in S) \subseteq \sqrt{\operatorname{jac} f_S} + (x_i; i \in S) \subset \sqrt{j_S}$$

and so $f \in \sqrt{I}$.

To prove that f is already in the integral closure $\overline{I}$ of I, we use the valuative criterion from above. To this end, let $\varphi\colon \mathbb{C}\{x_1, \dots, x_n\} \to V$ be a ring homomorphism into a discrete valuation ring with valuation $\mathbf{v}$. We need to show that $\varphi(f) \in \varphi(I)V$. We may assume that V is complete, in which case $V \cong K_t$ where $K \supseteq \mathbb{C}$ is a field extension. If $\varphi(I)V = V$ there is nothing to show. Otherwise $\varphi(I) \subseteq \mathfrak{m}_V$ and so $\varphi(f) \in \mathfrak{m}_V$, as $f \in \sqrt{I}$. By the chain rule

$$\frac{d\varphi(f)}{dt} = \sum_{i=1}^{m} \varphi\left(\frac{\partial f}{\partial x_i}\right) \frac{d\varphi_i}{dt}$$

where $\varphi_i := \varphi(X_i)$. Using $\mathbf{v}(d\varphi_i/dt) + 1 \geq \mathbf{v}(\varphi_i)$, we get

$$\begin{aligned}\mathbf{v}(\varphi(f)) &= \mathbf{v}\left(\frac{d\varphi(f)}{dt}\right) + 1 && \text{as } \varphi(f) \in \mathfrak{m}_V,\\ &\geq \min_i \left\{\mathbf{v}\left(\varphi\left(\frac{\partial f}{\partial x_i}\right)\right) + \mathbf{v}(\varphi_i)\right\}\\ &= \min_i \left\{\mathbf{v}\left(\varphi\left(x_i\frac{\partial f}{\partial x_i}\right)\right)\right\},\end{aligned}$$

so that $\varphi(f) \in \varphi(I)V$ as required. $\square$

Remark 1.1.9. Briançon-Skoda [**BSk**] proved that the integral closure of any ideal $J \subset \mathbb{C}\{x_1, \ldots, x_n\}$ satisfies $(\overline{J})^n \subseteq J$. They used this result in conjunction with the preceding proposition to conclude that $f^n \in (x_1(\partial f/\partial x_1), \ldots, x_n(\partial f/\partial x_n))$, thereby answering affirmatively a question raised earlier by J. Mather. As C. Huneke pointed out, there is presently no reasonable method known that would allow to obtain an explicit "Euler equation" of integral dependance for each f.

Now we finish the proof of theorem 1.1.1. Write $A = R/I$ with $R = \mathbb{C}\{x_1, \ldots, x_n\}$ and $I \subseteq \mathfrak{m}^2$. The associated Zariski-Jacobi sequence is

$$I/I^2 \xrightarrow{j} \Omega^1_{R/\mathbb{C}} \otimes_R A \longrightarrow \Omega^1_{A/\mathbb{C}} \to 0\,.$$

As $I \subseteq \mathfrak{m}^2$, the image of j is contained in $\mathfrak{m}\Omega^1_R \otimes_R A$. Dualizing the sequence into $\mathbb{C}$, claim 1.1.1(1) follows. If j mod $\mathfrak{m}$ is injective, then $\operatorname{Ext}^1_A\left(\Omega^1_A, \mathbb{C}\right) \cong \operatorname{Hom}_{\mathbb{C}}(I/\mathfrak{m}I, \mathbb{C})$ and so the dimension of this vector space is just $\mu(I)$, the number of equations defining A. In the crucial case that j mod $\mathfrak{m}$ is not injective, let $\int I \subseteq R$ be the ideal with $I \supseteq \int I \supseteq I^2$ and $\int I/I^2 = \ker j$. Thus $\int I$ consists of all functions f from I that have all their partial derivatives in I. By proposition 1.1.8, the ideal $\int I$ is integral over $\mathfrak{m}I$ and assertion 1.1.1(2) follows now from 1.1.5, in view of $\operatorname{Im} j = I/\int I$ and

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_A(\Omega^1_{A/K}, \mathbb{C}) = \mu(\operatorname{Im} j) = \dim_{\mathbb{C}} I/(\textstyle\int I + \mathfrak{m}I)\,. \quad \square$$

Remarks 1.1.10. (1) The notation $\int I$ used above is due to R. Pellikaan [**?**], [**Pel**] who calls this ideal the *primitive ideal* or *integral* of I. Colloquially, the result in 1.1.1(2) thus says that functions in the integral of I do not contribute to the codimension.

(2) The preceding proof shows that $\dim_{\mathbb{C}} \operatorname{Ext}^1_A(\Omega^1_{A/K}, \mathbb{C}) = \mu(I)$ iff $\int I \subseteq \mathfrak{m}I$. This happens for example if A is a complete intersection, that is, I is generated by a regular R-sequence, as then $s(I) = \mu(I)$ and each inequality in 1.1.1 becomes an equality. Conversely, if A is reduced and 1.1.1(2) becomes an equality, then A is a complete intersection by [**SSto**, (3.7)].

(3) Assume $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ has an isolated singularity at 0 and is not quasi-homogeneous with respect to any choice of coordinates. By K. Saito's theorem [**Sai**], then $f \notin \operatorname{jac}(f)$. For $I = (f) + \operatorname{jac}(f)$ this means $\int I \not\subseteq \mathfrak{m}I$ and so $A = \mathbb{C}\{x_1, \ldots, x_n\}/I$ satisfies

$$\dim A = \operatorname{emdim} A - \dim_{\mathbb{C}} \operatorname{Ext}^1_A(\Omega^1_{A/K}, \mathbb{C}) = 0 > \operatorname{emdim} A - \mu(I) = -1$$

whence here the lower bound in 1.1.1(2) is sharp and better than the trivial one.

It should be interesting to find other characterizations of those algebras A for which 1.1.1(2) becomes an equality.

1.2. Analytic algebras

Let k be a valued field and denote by $k\{X\}_n :=:= k\{X_1, \dots ,_n\}$ the ring of convergent power series. We recall the following definition, see [**GRe**].

Definition 1.2.1. A k-algebra A is called *analytic* if there is a finite k-algebra morphism $k\{X\}_n \to A$.

It follows that A is a semi-local k-algebra, i.e. A has at most a finite number of maximal ideals. One of the most important facts for analytic k-algebras is Weierstraß preparation theorem which we state in the elegant way given by Serre, see [**GRe**].

Theorem 1.2.2 (Preparation theorem). *Let $A \to B$ be a morphism of analytic k-algebras which is quasifinite, i.e. $B/\mathfrak{m}B$ is a finite dimensiona k-vector space for every maximal ideal $\mathfrak{m}$ of A. Then $A \to B$ is finite.*

We also remind the reader of the following useful facts, see loc.cit.

Remarks 1.2.3. 1. It is well known and a consequence of the preparation theorem that analytic k-algebras are noetherian.

2. If A is an analytic k-algebra with maximal ideals $\{\mathfrak{m}_1, \dots , \mathfrak{m}_t\}$, then $A \cong \prod_i A_{\mathfrak{m}_i}$.

3. If A is an analytic k-algebra and M is a finite module over A then we can form the trivial extension $A[M] := A \oplus M\varepsilon$ where the product is given by $(a + m\varepsilon)(a' + m'\varepsilon) := aa' + (am' + a'm)\varepsilon$, so that $\varepsilon^2 = 0$. It follows from the definition that $A[M]$ is again an analytic k-algebra. Note that $A \hookrightarrow A[M]$ is a subring and that $M\varepsilon \hookrightarrow A[M]$ is an ideal with $A[M]/M\varepsilon \cong A$.

1.2.4. Let A be a finitely generated k-algebra and $\mathfrak{m} \subseteq A$ a maximal ideal. Then we can associate to $A_\mathfrak{m}$ an analytic k-algebra $A^{an}_\mathfrak{m}$ in the following way. Assume that $a_1, \dots , a_n$ are k-algebra generators of A. As $k \to A/\mathfrak{m}$ is finite there are equations of integral dependence

$$f_i = a_i^{n_i} + c_{i1}a_i^{n_i-1} + \ldots + c_{in_i} \in \mathfrak{m}.$$

Then the map

$$k[X]_n := k[X_1, \dots , X_n] \to A \quad \text{with} \quad X_i \mapsto f_i$$

is finite, and $\mathfrak{m}$ contracts to the ideal $(X_1, \dots , X_n)$. We set

$$A^{an}_\mathfrak{m} := k\{X\}_n \otimes_{k[X]_n} A_\mathfrak{m},$$

which may be regarded as a subring of the completion

$$\hat{A}_\mathfrak{m} := k[[X]]_n \otimes_{k[X]_n} A_\mathfrak{m}.$$

This construction is independent of the choices made above as follows easily from the universal property which is as follows.

Proposition 1.2.5. *Let A be as above. Then every k-algebra morphism $\varphi : A_\mathfrak{m} \to B$ into an analytic k-algebra can be uniquely factored through $A^{an}_\mathfrak{m}$.*

The *proof* follows immediately from the construction.

There is an important special case of this construction. Let $E := \{X_1, \dots , X_n\}$ be a finite set and $\mathfrak{m} \subseteq k[E] := k[X_1, \dots , X_n]$ be a maximal ideal.

Definition 1.2.6. The k-algebra

$$k\{E\}_{\mathfrak{m}} := k[E]^{an}_{\mathfrak{m}}$$

will be called the *free analytic k-algebra on E at $\mathfrak{m}$.*

In the special that $k = \mathbb{C}$ or, more generally, that k is algebrically closed, every maximal ideal as above is of type $(X_1 - a_1, \ldots, X_n - a_n)$, with $a \in k^n$. Then $k\{E\}$ is just the ring of convergent power series

$$\sum_{\nu \in \mathbb{N}^n} c_\nu (X - a)^\nu, \quad c_\nu \in k.$$

Clearly this ring is isomorphic to $k\{X_1, \ldots, X_n\}$ via translation by a. But for an arbitrary field k the rings $k\{E\}_{\mathfrak{m}}$ are different, in general. In any case, this ring should be viewed as the ring of convergent power series near the point $a \hat{=} \mathfrak{m} \in \mathbb{A}^n_k$.

Proposition 1.2.7. *1. Let B be a local analytic k-algebra. Then there is a surjection $k\{E\}_{\mathfrak{m}} \to B$ for some E, $\mathfrak{m}$.*

2. Let $C \to B$ be a surjective morphism of analytic k-algebras. Then every k-algebra morphism $\beta : k\{E\}_{\mathfrak{m}} \to B$ can be lifted to a morphism $\gamma : k\{E\}_{\mathfrak{m}} \to C$.

Proof. For the proof of (1), consider generators $x_1, \ldots, x_m$ for $\mathfrak{m}_B$ and elements $x_{m+1}, \ldots, x_n \in B$ such that their residue classes generate $B/\mathfrak{m}_B$ over k. Set $E := \{x_1, \ldots, x_n\}$ and let $k[E] \to B$ be the k-algebra map induced by the inclusion $E \hookrightarrow B$. Let $\mathfrak{m} \subseteq k[E]$be the preimage of the maximal ideal of B. By 1.2.5 the map $k[E] \to B$ induces a morphism $p : k\{E\}_{\mathfrak{m}} \to B$. By construction $\mathfrak{m}B$ is the maximal ideal of B, and $k\{E\}_{\mathfrak{m}} \to B/\mathfrak{m}B$ is surjective. By the preparation theorem it follows that p is surjective.

In order to prove (2) we may assume that B, C are local. We first remark that there is a map $\tilde{\gamma} : k[E] \to C$ lifting the restriction $\beta | k[E]$. Then $\tilde{\gamma}^{-1}(\mathfrak{m}_C) = \mathfrak{m}$ necessarily, and so by 1.2.5 $\tilde{\gamma}$ induces a morphism of analytic k-algebras $\gamma : k\{E\}_{\mathfrak{m}} \to C$ lifting β. ☐

An important observation that in the category of analytic k-algebras there are always fibred coproducts also called analytic tensor products.

Proposition 1.2.8. *Let $A \to B$, $A \to C$ be morphisms of analytic k-algebras. Then there is an analytic tensor product $B \tilde{\otimes}_A C$.*

More explicitely, $B \tilde{\otimes}_A C$ is an analytic k-algebra together with morphisms $B \to B \tilde{\otimes}_A C$, $C \to B \tilde{\otimes}_A C$ such that the following hold.

- **(T)** For every pair of morphisms of A-algebras $\varphi : B \to D$, $\psi : C \to D$ there is a unique morphism $\varphi \otimes \psi : B \tilde{\otimes}_A C \to D$ restricting to φ, ψ on B, C respectively.

As for the usual tensor product, the maps $B \to B \tilde{\otimes}_A C$, $C \to B \tilde{\otimes}_A C$ are written as $b \mapsto b \otimes 1$, $c \mapsto 1 \otimes c$ respectively.

Proof. We will only give a sketch of the proof leaving the details to the reader. In the special case that $A = k$, $B = k\{E\}_{\mathfrak{m}}$, $C = k\{F\}_{\mathfrak{n}}$ we set

$$k\{E\}_{\mathfrak{m}} \tilde{\otimes}_k k\{F\}_{\mathfrak{n}} := \prod_{\mathfrak{M}} k\{E, F\}_{\mathfrak{M}},$$

where $\mathfrak{M}$ runs through all maximal ideals of $k[E, F]$ containing $\mathfrak{m}k[E, F]$ and $\mathfrak{n}k[E, F]$. It is easyly verified that this ring satisfies the universal property above.

In the case $A = k$, $B \cong k\{E\}_{\mathfrak{m}}/\mathfrak{b}$, $C \cong k\{F\}_{\mathfrak{n}}/\mathfrak{c}$ it is easily seen that the quotient of $k\{E\}_{\mathfrak{m}}\tilde{\otimes}_k k\{F\}_{\mathfrak{n}}$ module the ideal generated by $\mathfrak{b}$ and $\mathfrak{c}$ is an analytic tensor product. If $B \cong \prod B_i$, $C \cong \prod C_j$ are products of local analytic k-algebras then their tensor product is given by

$$B\tilde{\otimes}_k C := \prod_{i,j} B_i\tilde{\otimes}_k C_j.$$

Finaly, for a general analytic k-algebra A define $B\tilde{\otimes}_A B$ to be the quotient of $B\tilde{\otimes}_k C$ modulo the ideal generated by the elements $a \otimes 1 - 1 \otimes a$, $a \in A$. It is easy to check that this again satisfies the universal property, concluding the proof. ☐

Remarks 1.2.9. 1. If $A \to B$ is finite then $B \otimes_A C \cong B\tilde{\otimes}_A C$ by the universal property of the analytic tensor product.

2. The notion of analytic tensor product can be extended to modules. If M, N are modules over B, C respectively, then we set

$$M\tilde{\otimes}_A N := M \otimes_B (B\tilde{\otimes}_A C) \otimes_C N.$$

3. As with the usual tensor product the analytic tensor product behaves associative for products of three analytic k-algebras. We leave the straightforward formulation and its proof to the reader.

Finally we recall the notion of the differential module of an analytic algebra. For a morphism $A \to B$ of rings a universally finite A-derivation consists in an A-linear derivation

$$d : A \to \Omega_{B/A}$$

into a finite B-module $\Omega_{B/A}$ such that the following universal property is satisfied:

(D) If $\delta : B \to M$ is an A-derivation into a finite B-module then there is a unique B-linear map $h : \Omega_{B/A} \to M$ with $\delta = d \circ h$.

Proposition 1.2.10. *For any morphism of analytic k-algebras $A \to B$ there is a universally finite A-derivation $d : B \to \Omega_{B/A}$.*

Although this result is well known we will give here a prove which relies on a general principle and which will turn out to be very useful in later sections where we apply it to other classes of rings. Assume that $\mathbf{C}$ is a full subcategory of the category of (commutative) algebras over a fixed ring k satisfying the following two conditions.

(1) $\mathbf{C}$ admits fibered coproducts, denoted by $B\tilde{\otimes}_A C$ if $A \to B$, $A \to C$ are morphisms in $\mathbf{C}$.
(2) If $A \in \mathbf{C}$ and $\mathfrak{a} \subseteq A$ is an ideal then also $A/\mathfrak{a} \in \mathbf{C}$.

More generally as above we have the following result.

Proposition 1.2.11. *Let $A \to B$ be a morphism in $\mathbf{C}$ and set $\Omega_{B/A} := I/I^2$, where $I \subseteq B\tilde{\otimes}_A B$ is the kernel of the multiplication map $\mu : B\tilde{\otimes}_A B \to B$. Then the following hold.*

1. The map $d : B \to \Omega_{B/A}$ given by $d(b) = 1 \otimes b - b \otimes 1 \bmod I^2$, is an A–derivation.

2. This derivation satisfies the universal property for all derivations $\delta : B \to M$ into B–modules M such that $B[M] \in \mathbf{C}$. In other words, every such derivation factors through d via an B–linear map $h : \Omega_{B/A} \to M$.

Proof. The proof of (1) is a simple calculation which we leave to the reader. In order to show (2) we first note that I is generated by elements of the form $1\otimes b-b\otimes 1$, $b\in B$. In fact, if J be the ideal generated by all elements $1\otimes b-b\otimes 1$, $b\in B$ then $B\tilde{\otimes}_A B/J\in \mathbf{C}$, and this ring is isomorphic to $B\otimes_B B$ using the universal property of the tensor product. As $B\tilde{\otimes}_B B\cong B$ we get $I=J$.

In order to check the universal property for d, let $\delta : B\to M$ be an A-derivation into an A-module M. If $A[M]\in \mathbf{C}$ then $1-\delta : B\to B[M]$ is a morphism in $\mathbf{C}$ and so we get a map

$$(1-\delta)\tilde{\otimes}1 : B\tilde{\otimes}_A B\longrightarrow B[M].$$

Restricting to I on the left and projecting onto M on the right gives a B-linear map $h : I\to M$ satisfying $h(1\otimes b-b\otimes 1)=\delta(b)$. As I is generated by all elements $1\otimes b-b\otimes 1$, its square I^2 is generated by all products of such elements. The fact that δ is a derivation gives

$$\begin{aligned} h\left((1\otimes b-b\otimes 1)\left(1\otimes c-c\otimes 1\right)\right) &= h\left(1\otimes bc+bc\otimes 1-b\otimes c-c\otimes b\right)\\ &= \delta(bc)-b\delta(c)-c\delta(b)=0. \end{aligned}$$

Thus h induces a B-linear map $\bar{h} : I/I^2\to M$ satisfying $\bar{h}\circ d=\delta$. □

Remark 1.2.12. Note that the category of analytic k-algebras satisfies the conditions (1) and (2) above. Since trivial extensions of analytic algebras are again analytic algebras (see 1.2.3 (3)), the existence of a universally finite module of differentials is in fact a consequence of 1.2.11. Another example is given by the category of all finitely generated Λ-algebras, where Λ is a fixed ring. Later on we will see that also the category of Stein-algebras, i.e. algebras of type $\Gamma(K,\mathcal{O}_K)$, where K is a semianalytic compact Stein subset of a complex space X, will satisfy these conditions.

CHAPTER 2

Vector Fields and Extensions

2.1. Integration of vector fields

Nonvanishing vector fields can be integrated to exhibit locally product structures of complex spaces. This basic fact is well-known and appears in various forms in the literature, see e.g. [**Fis**, 2.12] for the complex analytic case, [**Mat**, 30.1] for the algebraic version, or [**Wal**, 3.2] in the differentiable case. We give a short proof of the result that we state in the algebraic as well as in the geometric setting, see 2.1.1, 2.1.2 and 2.1.5. We deduce from there an algebraic criterion as to whether a map of germs $(X, 0) \to (S, 0)$ of complex spaces is a product, that is, whether $(X, 0)$ is S-isomorphic to $(X_0 \times S, 0)$, cf. 2.1.9. To recognize such a product structure is central to many applications and later on we will formulate a quite more general criterion that decides the triviality of a deformation in terms of the vanishing of the Kodaira-Spencer class.

We begin with the local version that is apparently due to Zariski, cf. [**Tei**, p.586].

Proposition 2.1.1. (Integration of vector fields, algebraic form) *Let A be an analytic $\mathbb{C}$-algebra. If there exists a $\mathbb{C}$-derivation $\delta : A \to A$ such that $\delta(t) = 1$ for some $t \in \mathfrak{m}_A$, then $A_0 := \operatorname{Ker}\delta$ is an analytic subalgebra of A and the canonical map*

$$i : A_0\{T\} \longrightarrow A \quad \textit{with} \quad T \mapsto t$$

is an isomorphism of analytic algebras.

Proof. We consider the map

$$\begin{aligned} \varphi := \exp\left((T-t)\delta\right) : A &\longrightarrow A\{T\} \\ f &\longmapsto \sum_{\nu=0}^{\infty} \frac{\delta^{\nu}(f)}{\nu!}(T-t)^{\nu} \end{aligned}$$

that is a morphism of analytic $\mathbb{C}$-algebras, see [**GRe**]. First we show $\varphi(A) \subseteq A_0\{T\}$. For this, let $\tilde{\delta} : A\{T\} \to A\{T\}$ be the derivation with $\tilde{\delta}|A = \delta$ and $\tilde{\delta}(T) = 0$, so that $\operatorname{Ker}\tilde{\delta} = A_0\{T\}$. As $\tilde{\delta}(T - t) = -1$, it follows that

$$\tilde{\delta} \circ \varphi(f) = \sum_{\nu=0}^{\infty} \left[\frac{\delta^{\nu+1}(f)}{\nu!}(T-t)^{\nu} + \frac{\delta^{\nu}(f)}{\nu!}\tilde{\delta}\left((T-t)^{\nu}\right) \right] = 0$$

and we get $\varphi : A \to A_0\{T\}$ as claimed. Now we show that $\varphi \circ i = id_{A_0\{T\}}$. Indeed, if $f = \sum a_\nu t^\nu = i(\sum a_\nu T^\nu)$ with $a_\nu \in A_0$, then

$$\varphi(f) = \sum_{\nu} \varphi(a_\nu)\varphi(t)^{\nu} = \sum a_\nu T^\nu$$

as, by definition, $\varphi(a) = a$ for every $a \in A_0$ and $\varphi(t) = T$. Finally we prove that φ is injective on A. Assume that $f \in \operatorname{Ker} \varphi$ and consider a representation $f = t^k \tilde{f}$ for some $k \geq 0$. That $0 = \varphi(f) = \varphi\left(t^k \tilde{f}\right) = T^k \varphi(\tilde{f})$ implies

$$0 = \varphi(\tilde{f}) = \sum_{\nu=0}^{\infty} \frac{\delta^\nu(\tilde{f})}{\nu!}(T - t)^\nu \equiv \tilde{f} \bmod (T - t) \ .$$

But $\tilde{f} \in A$ does not depend upon T, hence setting $T = 0$ shows $\tilde{f} \in tA$, whence $f \in t^{k+1}A$. It follows that $\operatorname{Ker} \varphi \subseteq \bigcap_k t^k A = 0$. ☐

Corollary 2.1.2 (Jacobian criterion). *Let* $\delta_1, \ldots, \delta_n$ *be* $\mathbb{C}$*-derivations on an analytic* $\mathbb{C}$*-algebra* A *such that* $\det(\delta_i(t_j)_{1\leq i,j\leq n})$ *is a unit in* A *for some elements* $t_1, \ldots, t_n$ *in* $\mathfrak{m}_A$*. There is then an isomorphism* $\varphi : A_0\{T_1, \ldots, T_n\} \to A$ *for some analytic* $\mathbb{C}$*-algebra* A_0*.*

Proof. We proceed by induction on n. In case $n = 0$, there is nothing to show. In the general case, after renumbering the δ_i, we may assume that $\delta_1(t_1)$ is a unit in A. Multiplying δ_1 by a suitable element of A we may further assume that $\delta_1(t_1) = 1$. Using 2.1.1 we get an isomorphism $A \cong B\{T_1\}$ where $B := \operatorname{Ker} \delta_1$. This isomorphism identifies δ_1 with the derivation $\partial/\partial T_1$ of $B\{T_1\}$, and $t_1 \in A$ with $T_1 \in B\{T_1\}$. Replacing δ_i by $\delta_i' := \delta_i - \delta_i(t_1)\delta_1$ for $i = 2, \ldots, n$, we may assume that $\delta_i(t_1) = 0$ for $i \geq 2$. These δ_i, $i \geq 2$, define derivations, say $\bar{\delta}_i$, on $B \cong A/(t_1)$. By construction, $\det(\bar{\delta}_i(\bar{t}_j)_{2\leq i,j\leq n})$ is a unit in B where $\bar{t}_i$ denotes the residue class of t_i in B. Using the induction hypothesis the result follows. ☐

The proof shows that if $\delta_i(t_j) = \delta_{i,j}$, then there is even an isomorphism φ as above that identifies t_i with T_i and δ_i with $\partial/\partial T_i$ for each i.

This result implies easily the following smoothness criterion for a morphism that we formulate again in the language of analytic algebras.

Corollary 2.1.3. *Let* $A \to B$ *be a morphism of analytic* $\mathbb{C}$*-algebras such that* $\Omega^1_{B/A}$ *is a free* B*-module, say of rank* n*. There exists then an isomorphism of* A*-algebras* $B \cong (A/\mathfrak{a})\{T_1, \ldots, T_n\}$ *for some ideal* $\mathfrak{a}$ *of* A*. If in addition* $A \to B$ *is injective then* $A \to B$ *is smooth.*

Proof. We choose $t_1, \ldots, t_n$ in the maximal ideal $\mathfrak{m}_B$ of B such that the differentials $dt_1, \ldots, dt_n$ form a basis of $\Omega^1_{B/A}$. Let $\delta_1, \ldots, \delta_n$ be the dual basis considered as A-derivations of B. This means that $\delta_i(t_j) = \delta_{i,j}$ for $1 \leq i, j \leq n$. By 2.1.2 and the above remark, there is an analytic $\mathbb{C}$-algebra C and an isomorphism $\varphi : C\{T_1, \ldots, T_n\} \to B$ that identifies t_i with T_i and δ_i with $\partial/\partial T_i$. As $\delta_1, \ldots, \delta_n$ are A-derivations, the image of A in B is contained in C that we consider as a subring of B via φ. Since $dt_1, \ldots, dt_n$ generate $\Omega^1_{B/A}$, the Zariski-Jacobi sequence shows that $\Omega^1_{C/A} = 0$. Hence $A \to C$ is surjective. This proves the first part and the second part is an immediate consequence of it. ☐

Remarks 2.1.4. (1) The preceding results can be generalized to include trivialization of modules. In the situation of 2.1.1, assume that M is a finite A-module carrying a covariant derivative $\nabla : M \to M$ with respect to δ, that means

$$\nabla(fm) = \delta(f)m + f\nabla(m)$$

for $f \in A$, $m \in M$. There is then a trivialization $M \cong M_0\{T\}$ over φ where $M_0 = \operatorname{Ker}\nabla$ is a finite A_0-module. In fact, $B := A \times M$ with $(a,m)(b,n) = (ab, an + bm)$ is an analytic A-algebra and $\delta \times \nabla : B \to B$ is a $\mathbb{C}$-derivation that maps $(t,0)$ to $1 = (1,0)$ in B and whose kernel is $\operatorname{Ker}(\delta \times \nabla) = A_0 \times M_0 =: B_0$. Therefore, by 2.1.1, $B \cong B_0\{T\}$ and so $M \cong M_0\{T\}$.

(2) The same argument allows it to generalize 2.1.2 to modules. We leave the straightforward formulation to the reader.

(3) The above results hold more generally for analytic algebras over an arbitrary valued field of characteristic 0 as the proofs show. In particular, they apply to complete local K-algebras over a field of characteristic 0.

Reformulating 2.1.1 in geometric terms gives the following proposition.

Proposition 2.1.5. (Integration of vector fields, geometric form) *Let $X \to \Sigma$ be a morphism of complex spaces and let $t \in \Gamma(X, \mathcal{O}_X)$ be a function. Set $X_0 = \{t = 0\}$ and assume that there is a Σ-derivation $\delta : \mathcal{O}_X \to \mathcal{O}_X$ with $\delta(t) = 1$. There is then a neighbourhood U of $X_0 \times \{0\}$ in $X_0 \times \mathbb{C}$ that fits into a diagram*

$$\begin{tikzcd}
X_0 \times \mathbb{C} \supseteq & U \arrow[r, hook, "i"] & X \\
& X_0 \times \{0\} \arrow[u, hook] \arrow[r, "\cong"] & X_0, \arrow[u, hook]
\end{tikzcd}$$

where i is an open Σ-embedding. One can take $i = \exp((T - t)\delta)$, with T the coordinate on $\mathbb{C}$.

Proof. Identifying X_0 with $X_0 \times \{0\}$ and applying 2.1.1 to every point of X_0 yields an isomorphism of sheaves

$$\mathcal{O}_X|X_0 \xrightarrow{\cong} \mathcal{O}_{X_0 \times \mathbb{C}}|X_0 \times \{0\}$$

given by $\exp((T - t)\delta)$, where $\mathcal{O}_X|X_0$ is the topological restriction of $\mathcal{O}_X$ to X_0 and $\mathcal{O}_{X_0 \times \mathbb{C}}|X_0 \times \{0\}$ is the corresponding restriction to $X_0 \times \{0\}$. This defines an isomorphism of germs

$$(X, X_0) \xrightarrow{\sim} (X_0 \times \mathbb{C}, X_0 \times \{0\})$$

thus proving 2.1.5. □

Remark 2.1.6. The reader is encouraged to reformulate also 2.1.3 and 2.1.4 (1) in geometric terms.

In applications, a relative version of 2.1.5, or, algebraically, of 2.1.1, is often used.

Corollary 2.1.7. *Let $f : X \to S$ be a morphism of complex spaces over Σ. Assume that there is a* pair of compatible vector fields $(D, \delta) \in \operatorname{Der}_\Sigma(\mathcal{O}_X, \mathcal{O}_X) \times \operatorname{Der}_\Sigma(\mathcal{O}_S, \mathcal{O}_S)$*; meaning that*

$$\begin{tikzcd}
f^{-1}(\mathcal{O}_S) \arrow[r, "f^{-1}(\delta)"] \arrow[d] & f^{-1}(\mathcal{O}_S) \arrow[d] \\
\mathcal{O}_X \arrow[r, "D"] & \mathcal{O}_X
\end{tikzcd}$$

commutes; and that there is a section $t \in \Gamma(S, \mathcal{O}_S)$ *satisfying* $\delta(t) = 1$. *With* $X_0 = \{f^*(t) = 0\}$ *and* $S_0 = \{t = 0\}$, *there are then open neighbourhoods* $U \subseteq X_0 \times \mathbb{C}$ *of* $X_0 \times \{0\}$, *and* $V \subseteq S_0 \times \mathbb{C}$ *of* $S_0 \times \{0\}$ *resp. that fit into a diagram*

$$\begin{tikzcd}
X_0 \times \mathbb{C} \arrow[d, "f_0 \times id_{\mathbb{C}}"'] & U \arrow[l, hook'] \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "f"] \\
S_0 \times \mathbb{C} & V \arrow[l, hook'] \arrow[r, hook, "j"] & X
\end{tikzcd}$$

with open embeddings i, j *over* Σ *such that*

$$i|X_0 \times \{0\} : X_0 \hookrightarrow X \quad \text{and} \quad j|S_0 \times \{0\} : S_0 \hookrightarrow S$$

are the given inclusions.

Proof. The rather longwinded formulation simply means that integrating a compatible pair of derivations results in a compatible pair of trivializations. Just take $i = \exp((T - f^*t)D)$ and $j = \exp((T - t)\delta)$. $\square$

If t and δ are given, so that S is already known to be locally a product, the question whether the map f respects the product structure depends thus only upon the existence of a vector field D on X that lifts δ. In the typical case where S is an open neighbourhood of a point $0 \in \mathbb{C}$, so that $f : X \to S$ represents a one parameter family of spaces, the family is analytically trivial over S iff there exists a vector field on X that lifts $\partial/\partial t$ where t is a coordinate on S.

One thus wants criteria that ensure liftability of vector fields. We formulate first the result for the germ of a map to $(\mathbb{C}, 0)$. To do so, and for further use in the following section, we introduce the notion of the Jacobian submodule of a map of germs.

2.1.8. Let $(S, 0)$ be the germ of a complex space and $g_1, \dots g_r \in \mathcal{O}_{\mathbb{C}^N \times S, 0}$ holomorphic functions defined on some neighbourhood of $0 = (0, 0)$ in $\mathbb{C}^N \times S$ that vanish at the distinguished point. We equip $\mathbb{C}^N$ with coordinates $(z_1, \dots, z_N)$ and denote by I the ideal of $\mathcal{O}_{\mathbb{C}^N \times S, 0}$ that is generated by $g = (g_1, \dots, g_r)$. Let

$$Jg : \mathcal{O}^N_{\mathbb{C}^N \times S, 0} \longrightarrow \mathcal{O}^r_{\mathbb{C}^N \times S, 0}$$

be the map given by the Jacobi matrix

$$Jg = \left(\frac{\partial g_\varrho}{\partial z_\nu} \right), \qquad 1 \le \varrho \le r,\ 1 \le \nu \le N\ .$$

The image of this map is called the *Jacobian module* of g and denoted by $\mathrm{jac}_S(g)$. The *extended Jacobian module* of g is

$$\mathrm{jac}^e_S(g) := \sum_i g_i \mathcal{O}^r_{\mathbb{C}^N \times S, 0} + \mathrm{jac}_S(g) \subseteq \mathcal{O}^r_{\mathbb{C}^N \times S, 0}\ ,$$

the submodule generated by the image of the Jacobi matrix together with $I\mathcal{O}^r_{\mathbb{C}^N \times S, 0}$. If S is just a reduced point, we write $\mathrm{jac}(g)$, $\mathrm{jac}^e(g)$ instead of $\mathrm{jac}_S(g)$, $\mathrm{jac}^e_S(g)$.

In the simplest case S is a reduced point and $r = 1$. Thus $g := g_1$ is a single function and $\mathrm{jac}(g)$ is its usual *Jacobian ideal* $(\partial g/\partial z_1, \dots, \partial g/\partial z_N)$, whereas the *extended Jacobian ideal* is $\mathrm{jac}^e(g) = (g, \partial g/\partial z_1, \dots, \partial g/\partial z_N)$, the ideal that describes the singular locus of $X = (g^{-1}(0), 0) \subset (\mathbb{C}^N, 0)$.

We now come to the promised first application of 2.1.5. Let

$$f : (X, 0) \longrightarrow (\mathbb{C}, 0) =: (S, 0)$$

be the germ of a holomorphic map. We may assume that $(X, 0)$ is embedded into $(\mathbb{C}^N \times \mathbb{C}, 0)$ in such a way that f is induced by the projection onto the second factor. Let I be the ideal of $(X, 0)$ in $\mathcal{O}_{\mathbb{C}^N \times S,0}$, assume that $g_1, \dots, g_r \in \mathcal{O}_{\mathbb{C}^N \times S,0}$ generate I and choose coordinates $(z_1, \dots, z_N, t)$ on $\mathbb{C}^N \times S$.

The important infinitesimal criterion for the triviality of such a map can now be stated as follows.

Theorem 2.1.9. (Product criterion) *With the notations just introduce, the following are equivalent.*

(1) *There is a vector field* $D \in \mathrm{Der}(\mathcal{O}_{X,0}, \mathcal{O}_{X,0})$ *lifting* $\partial/\partial t \in \mathrm{Der}(\mathcal{O}_{S,0}, \mathcal{O}_{S,0})$, *that is*

$$\begin{tikzcd}
\mathcal{O}_{S,0}) \arrow[r, "\partial/\partial t"] \arrow[d, "f^*"'] & f^{-1}(\mathcal{O}_{S,0}) \arrow[d, "f^*"'] \\
\mathcal{O}_{X,0} \arrow[r, "D"] & \mathcal{O}_{X,0}
\end{tikzcd}$$

commutes.

(2) $(\partial g_1/\partial t, \dots, \partial g_r/\partial t) \in \mathrm{jac}^e_S(g) \subseteq \mathcal{O}^r_{\mathbb{C}^N \times S,0}$.

(3) $(X, 0)$ *is* S*-isomorphic to* $(X_0 \times S, 0)$ *where* X_0 *is the fibre* $f^{-1}(0)$ *of* f.

Proof. A vector field D on X is given by a linear combination

$$D = \alpha \frac{\partial}{\partial t} - \Sigma \beta_i \frac{\partial}{\partial z_i} \in \mathrm{Der}\left(\mathcal{O}_{\mathbb{C}^{N+1},0}, \mathcal{O}_{\mathbb{C}^{N+1},0}\right) \qquad (*)$$

such that

$$D(g_1) = \cdots = D(g_r) \equiv 0 \bmod I \ . \qquad (**)$$

This vector field lifts $\partial/\partial t$ iff $\alpha \equiv 1 \bmod I$. In particular, if there is a lifting then there is one with $\alpha = 1$. Writing out $(**)$ for $\alpha = 1$ gives (2). Conversely, if (2) is satisfied then there are $\beta_i \in \mathcal{O}_{\mathbb{C}^{N+1},0}$ such that

$$\frac{\partial(g_1, \dots, g_r)}{\partial t} \equiv \sum_{i=1}^{N} \beta_i \frac{\partial(g_1, \dots, g_r)}{\partial z_i} \mod I$$

and the corresponding vector field D given by $(*)$ with $\alpha = 1$ satisfies $(**)$. Thus we have shown that (2) is equivalent to (1), and the equivalence of (1) and (3) follows from 2.1.7. □

We will sometimes need a variant of 2.1.9 where $f : (X, 0) \to (S, 0)$ admits a section and one wants a trivialization along that section. More precisely, embedding $(X, 0)$ appropriately, one may assume that $0 \times S \subseteq X \subseteq \mathbb{C}^n \times S$ with f again induced by the second projection. The question then becomes whether there exists a trivialization $(X, 0) \to (X_0 \times S, 0)$ that is the identity on $0 \times S$.

Proposition 2.1.10. *In the situation just described the following are equivalent.*

(1) *There is an* S*-isomorphism* $(X, 0) \to (X_0 \times S, 0)$ *inducing the identity on* $0 \times S$.

(2) $(\partial g_1/\partial t, \dots, \partial g_r/\partial t) \in I\mathcal{O}^r_{\mathbb{C}^N \times S,0} + \mathfrak{m}\,\mathrm{jac}_S(g)$ *where* $\mathfrak{m}$ *is the maximal ideal of* $\mathcal{O}_{\mathbb{C}^N,0}$.

PROOF. Let D be a lifting of the vector field $\partial/\partial t$. Integrating D gives an isomorphism preserving $0 \times S$ iff D is tangent to $0 \times S$ which just means that $\beta_i \in \mathfrak{m}\mathcal{O}_{\mathbb{C}^N \times S,0}$ in the notations of the proof of 2.1.9. The rest follows as before. □

Later on we will interpret condition 2.1.9 (2) as vanishing of the Kodaira-Spencer class with respect to contact equivalence for the unfolding

$$g : \left(\mathbb{C}^N \times \mathbb{C}, 0\right) \to (\mathbb{C}^r \times \mathbb{C}, 0)$$

of $g(z, 0)$, see Sect. 3.2. Indeed, it is a rather general principle that vanishing of the corresponding Kodaira-Spencer class implies triviality of a deformation.

Our final example in this section highlights some of the finer points, in particular the analytic nature of the results.

2.1.11. (A family of elliptic curves) Consider the complex projective plane $\mathbb{P}^2$ with homogeneous coordinates (x, y, z), and in $\mathbb{P}^2 \times \mathbb{C}$ over $\mathbb{C}$ the family of plane cubics

$$X = \{y^2 z - 4x^3 + xz^2 + \frac{t}{3\sqrt{3}} z^3 = 0\} \subseteq \mathbb{P}^2 \times \mathbb{C}$$

where t is the coordinate in the base space $S = \mathbb{C}$. The cubics are in Weierstraß normal form with $g_2 = 1$ and $g_3 = t/(3\sqrt{3})$, whence the discriminant equals $\Delta = g_2^3 - 27g_3^2 = 1 - t^2$ and the j-invariant is $j = 1728 g_2^3/\Delta = 1728/(1 - t^2)$. As the j-invariant is nowhere constant, the family is not trivial near any point in $\mathbb{C}$.

Now take away the line at infinity, $(z = 0) \subset \mathbb{P}^2$, to obtain the family of punctured cubics

$$X' = \{y^2 - 4x^3 + x + \frac{t}{3\sqrt{3}} = 0\} \subseteq \mathbb{C}^2 \times \mathbb{C}\ .$$

With $g = y^2 - 4x^3 + x + \frac{t}{3\sqrt{3}}$ one verifies easily the identity

$$(1 - t^2)\frac{\partial g}{\partial t} + (6x^2 - \sqrt{3}tx - 1)\frac{\partial g}{\partial x} + \frac{1}{2}y(18x - 3\sqrt{3}t)\frac{\partial g}{\partial y} = (18x - 3\sqrt{3}t)g$$

whence the vector field

$$D = \frac{\partial g}{\partial t} + \frac{1}{1 - t^2}\left((6x^2 - \sqrt{3}tx - 1)\frac{\partial}{\partial x} + \frac{1}{2}y(18x - 3\sqrt{3}t)\frac{\partial}{\partial y}\right)$$

is analytic on X' over $\{\Delta \neq 0\} = \mathbb{C} \setminus \{\pm 1\}$ and lifts there $\partial/\partial t$. Thus locally the family is analytically trivial over $\mathbb{C} \setminus \{\pm 1\}$. Note that no local trivialization can include either point $t = \pm 1$, the fibres there are nodal cubics that are already topologically different from the other fibres.

The indicated vectorfield D is algebraic, but its integral $\exp((T - t)D)$ is definitely not. Indeed, there cannot be an algebraic trivialization, that is one given by rational functions, over any open subset of $\mathbb{C}$: Such a trivialization would induce isomorphisms between the rational function fields of the smooth fibres, but their isomorphism types are distinguished precisely by the j-invariant! The following pictures show part of the real points of X' over $-1 \leq t \leq 1$ and the vector field D on that surface.

2.2. An application: Contact equivalence of mapping germs

In this section we consider the question when two functions, or, more generally, two r-tuples of functions, define isomorphic germs of complex spaces. The corresponding equivalence relation is called *contact equivalence* and was first introduced and studied by Mather in a series of remarkable papers, [**Math**]. The main result here is 2.2.2 that gives a sufficient criterion for the contact equivalence of mapping germs in terms of their Jacobian modules. Specializing to hypersurface singularities, one is led to the notion of *finite determinacy* of the germ of a function up to contact equivalence. After treating the sufficient criterion by Mather-Tougeron **??** for k-determinacy in 2.2.5, we prove in 2.2.9 that isolated hypersurface singularities are finitely determined, the colength of the Jacobian ideal giving a coarse upper bound for the determinacy, 2.2.6(2).

Definition 2.2.1 (Mather). Two r-tuples $f, \tilde{f} \in \mathcal{O}^r_{\mathbb{C}^N,0}$ of holomorphic functions with $f(0) = \tilde{f}(0) = 0$ are called *contact equivalent* if the zero sets

$$X := \{f_1 = \cdots = f_r = 0\} \quad \text{and} \quad Y =: \{\tilde{f}_1, = \cdots = \tilde{f}_r = 0\}$$

define isomorphic germs of analytic spaces.

Equivalently there is an analytic isomorphism $\varphi : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ such that

$$f \circ \varphi = \varepsilon \tilde{f}$$

for some invertible $r \times r$-matrix of holomorphic functions $\varepsilon = (\varepsilon_{ij})_{i \le i,j \le r}$. To indicate that equivalent tuples have isomorphic *vanishing sets*, this relation is also called V-equivalence in [**?**], [**AVGL**]; whereas in [**?**] it is named $\mathcal{K}$-equivalence.

The next theorem, due to Mather, see [**?**], is the central result of this section. It uses the notion of the Jacobian ideal as introduced in 2.1.8.

Theorem 2.2.2. *Let $(X_0, 0) \subseteq (\mathbb{C}^N, 0)$ be a germ defined by $f_1, \ldots, f_r \in \mathcal{O}_{\mathbb{C}^N,0}$. Assume that $g_1, \ldots, g_r \in \mathcal{O}_{\mathbb{C}^N,0}$ are functions such that*

$$\sum_{\varrho=1}^{r} g_\varrho \mathcal{O}^r_{\mathbb{C}^N,0} + \mathfrak{m}\,\mathrm{jac}(g) \subseteq \mathfrak{m} \left(\sum_{\varrho=1}^{r} f_\varrho \mathcal{O}^r_{\mathbb{C}^N,0} + \mathfrak{m}\,\mathrm{jac}(f) \right)$$

where $\mathfrak{m} \subseteq \mathcal{O}_{\mathbb{C}^N,0}$ is the maximal ideal. Then the r-tuples f and $f + g$ are contact equivalent, the germ $(X_0, 0)$ is isomorphic to the germ $(X_1, 0)$ defined by the equations $f_\varrho + g_\varrho = 0$, $1 \le \varrho \le r$, in $(\mathbb{C}^N, 0)$.

Proof. The stated condition implies first of all that $g_\varrho(0) = 0$ for $1 \le \varrho \le r$. Now consider the functions $F_\varrho(x, t) := f_\varrho(x) + t g_\varrho(x)$ on $\mathbb{C}^N \times S$ where t is the coordinate on $S := \mathbb{C}$. Let $X \subseteq \mathbb{C}^N \times \mathbb{C}$ be the set of zeros of $F_1, \ldots, F_r$. As $F(0, t) = 0$, there is some open neighbourhood of $\{0\} \times \mathbb{C}$ in which X is a closed subspace defined by the ideal sheaf $\mathcal{J} = (F_1, \ldots, F_r)$. Denote by $X_\tau := \pi^{-1}(\tau)$ the fibre over $\tau \in \mathbb{C}$. We will show that the map germ

$$\pi : (X, (0, \tau)) \longrightarrow (\mathbb{C}, \tau)$$

given by the projection onto the last factor is trivial along $0 \times (\mathbb{C}, \tau)$ for every $\tau \in \mathbb{C}$ in the following sense: on a neighbourhood of $(0, \tau)$ there is an isomorphism $\Phi : X \to X_\tau \times \mathbb{C}$ over $(\mathbb{C}, \tau)$ that preserves $0 \times \mathbb{C}$. In particular, for each $\tau \in \mathbb{C}$, one gets $(X_{\tau'}, 0) \cong (X_\tau, 0)$ for all τ' in a neighbourhood of τ, whence all r-tuples

$F(x,\tau)$ are contact equivalent, especially those at $\tau=0$ and at $\tau=1$. For the existence of Φ, it suffices by 2.1.10 to show that

$$(1)\quad g=(\partial F_1/\partial t,\dots,\partial F_r/\partial t)\in M_\tau(F):=\sum_\varrho F_\varrho\mathcal{O}^r_{\mathbb{C}^{N+1},(0,\tau)}+\mathfrak{m}\,\mathrm{jac}_S(F)_{(0,\tau)}\ .$$

Obviously (1) is implied by

$$(2)\qquad M_\tau(F)=M_\tau(f_{\mathbb{C}}):=\sum_\varrho f_\varrho\mathcal{O}^r_{\mathbb{C}^{N+1},(0,\tau)}+\mathfrak{m}\,\mathrm{jac}(f)\mathcal{O}_{\mathbb{C}^{N+1},(0,\tau)}$$

as g is contained in the right hand side of (2). To prove (2), observe that the r-tuples

$$F-f=tg\quad\text{and}\quad z_\nu(\frac{\partial F}{\partial z_i}-\frac{\partial f}{\partial z_i})=z_\nu t\frac{\partial g}{\partial z_i},\quad 1\le\nu\le N,$$

are in the submodule $\mathfrak{m}M_\tau(f_{\mathbb{C}})$ by assumption. These equations state that $M_\tau(f_{\mathbb{C}})$ is contained in $M_\tau(F)+\mathfrak{m}M_\tau(f_{\mathbb{C}})$, but then $M_\tau(F)$ and $M_\tau(f_{\mathbb{C}})$ are already equal by the lemma of Nakayama. □

In applications, conditions stronger than the stated one are often satisfied, for instance, $(g)\subseteq\mathfrak{m}(f)$. For germs of functions, that is in the special case $r=1$, the theorem reads as follows.

Corollary 2.2.3. *If $f,g\in\mathcal{O}_{\mathbb{C}^N,0}$ are functions such that*

$$(g)+\mathfrak{m}\,\mathrm{jac}(g)\subseteq f\mathfrak{m}+\mathfrak{m}^2\,\mathrm{jac}(f)$$

then g and $g+h$ define isomorphic germs of hypersurfaces. □

As an elementary example consider the function $f:=z_1^2+\dots+z_N^2$. Its Jacobian ideal is just the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{(\mathbb{C}^N,0)}$ and so for any $g\in\mathfrak{m}^3$, the functions f and $f+g$ define isomorphic singularities.

Definition 2.2.4. A function $f\in\mathcal{O}_{\mathbb{C}^N,0}$ is called *k-determined* (with respect to contact equivalence) if for all $g\in\mathfrak{m}^{k+1}$ the functions f and $f+g$ are contact equivalent. A function is said to be *finitely determined* if it is k-determined for some k. The smallest such k is the *degree of determinacy* of f.

If f is k-determined then f is already determined by its so called *k-jet*

$$j^kf=\sum_{\substack{\nu\in\mathbb{N}^N\\ \nu_1+\dots+\nu_N\le k}}f_\nu z^\nu,$$

where $\sum f_\nu z^\nu$ is the Taylor series of f. In particular, a finitely determined function is always contact equivalent to a polynomial and so defines an algebraic hypersurface singularity.

Corollary 2.2.5 (Mather-Tougeron**??**). *If $f\in\mathcal{O}_{\mathbb{C}^N,0}$ satisfies*

$$\mathfrak{m}^{k+1}\subseteq\mathfrak{m}\left((f)+\mathfrak{m}\,\mathrm{jac}(f)\right)$$

then f is k-determined.

Proof. If $g\in\mathfrak{m}^{k+1}$ then also $(g)+\mathfrak{m}\,\mathrm{jac}(g)\subseteq\mathfrak{m}^{k+1}$, whence the assertion follows from 2.2.3. □

REMARKS 2.2.6. (1) If $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^N,0}$ are contact equivalent then the analytic algebras $\mathcal{O}_{\mathbb{C}^N,0}/(f_1)$ and $\mathcal{O}_{\mathbb{C}^N,0}/(f_2)$ are isomorphic. In particular, the algebras $\mathcal{O}_{\mathbb{C}^N,0}/\operatorname{jac}^e(f_1)$ and $\mathcal{O}_{\mathbb{C}^N,0}/\operatorname{jac}^e(f_2)$ are isomorphic, and so are $\mathcal{O}_{\mathbb{C}^N,0}/\operatorname{jac}(f_1)$ and $\mathcal{O}_{\mathbb{C}^N,0}/\operatorname{jac}(f_2)$.

The corresponding dimensions as $\mathbb{C}$-vector spaces,

$$\mu(f) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^N,0}/\operatorname{jac}(f), \quad \text{the } \textit{Milnor number} \text{ of } f$$

and

$$\tau(f) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^N,0}/\operatorname{jac}^e(f), \quad \text{the } \textit{Tyurina number} \text{ of } f$$

are important invariants of the singularity defined by f that are usually easier to determine than the order of determinacy.

(2) Assume that f is of *multiplicity* at least k, that is $f \in \mathfrak{m}^k$. If the dimension

$$\ell = \dim_{\mathbb{C}} \mathfrak{m}^{k+1}/\mathfrak{m}\left((f) + \mathfrak{m}\operatorname{jac}(f)\right)$$

is finite then f is $(k+\ell)$-determined. This follows from 2.2.5 as every $\mathcal{O}_{\mathbb{C}^N,0}$-module of length ℓ is annihilated by $\mathfrak{m}^\ell$. The example

$$f := z_1^2 + \ldots + z_{N-1}^2 + z_N^{\ell+2}$$

shows that f is in general not $k+\ell-1$-determined. It can be shown however that this is essentially the only example where $\mathfrak{m}^{k+1}/\mathfrak{m}((f) + \operatorname{jac}(f))$ is of dimension ℓ but f is not $k+\ell-1$-determined.

(3) As a special case of 2.2.5 it follows that the germ $\{f = 0\}$ is smooth in a neighbourhood of 0 iff f is 1-determined. Clearly this is a reformulation of the implicit function theorem.

(4) The condition in 2.2.3 is not necessary for f and $f+g$ to be contact equivalent, take for example $f = g$.

For a function that defines an isolated singularity and is a homogeneous polynomial on $\mathbb{C}^N$ in suitable coordinates, the degree of determinacy is easy to obtain.

PROPOSITION 2.2.7. *Assume that* $(N-1)(d-2) \geq 2$. *If* $f \in \mathbb{C}[z_1, \ldots, z_N]$ *is a homogeneous polynomial of degree* d *with isolated singularity at* 0 *then its degree of determinacy equals* $N(d-2)$.

PROOF. We begin by showing that f is $N(d-2)$-determined. The Euler identity $d \cdot f = \sum_{i=1}^N z_i \partial f/\partial z_i$ shows that $f \in \mathfrak{m}\operatorname{jac}(f)$, and thus 3.1.6 yields the claim as soon as we know that

$$(*) \qquad \mathfrak{m}^{N(d-2)+1} \subseteq \mathfrak{m}^2 \operatorname{jac}(f).$$

Consider first the special case of the polynomial $f = z_1^d + \ldots + z_N^d$, whose Jacobi ideal $\operatorname{jac}(f)$ is generated by the monomials $z_1^{d-1}, \ldots, z_N^{d-1}$. As every monomial of degree $N(d-2)+1$ contains at least one factor of the form z_i^{d-1} and as $N(d-2)+1-(d-1) = (N-1)(d-2) \geq 2$, we obtain $(*)$ for this example.

Indeed a single example suffices to establish the general case: Consider the family of all homogeneous polynomials of degree d in N variables,

$$F(z,s) := \sum_{\substack{\nu \in \mathbb{N}^N \\ |\nu| = d}} s_\nu z^\nu$$

defined on $\mathbb{C}^N \times \mathbb{C}^k$ with $k := \binom{N+d-1}{d}$. Let $U \subseteq \mathbb{C}^k$ denote the open subset corresponding to all homogeneous polynomials that define an isolated singularity. Denoting by $\mathfrak{m}_N \subseteq \mathcal{O}_{\mathbb{C}^N}$ the ideal sheaf of the origin, the algebra

$$\mathcal{A} := \mathcal{O}_{\mathbb{C}^N \times U}/\mathfrak{m}_N^2(\partial F/\partial z_1, \dots, \partial F/\partial z_N)$$

is a coherent graded sheaf of algebras over U. At every point $s \in U$ its fibre $\mathcal{A}/\mathfrak{m}_{U,s}\mathcal{A}$ is just the $\mathbb{C}$-algebra $\mathcal{O}_{\mathbb{C}^N,0}/\mathfrak{m}_N^2 \operatorname{jac}(f_s)$ where $f_s(z) := F(z, s)$. Using Bezout's theorem and the fact that the generators of $\operatorname{jac}(f_s)$ form a regular sequence, it follows that the dimension of this algebra is independent of s, given by

$$\begin{aligned}\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^N,0}/\mathfrak{m}_N^2 \operatorname{jac}(f_s) &= \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^N,0}/\operatorname{jac}(f_s) + \dim_{\mathbb{C}} \operatorname{jac}(f_s)/\mathfrak{m}_N^2 \operatorname{jac}(f_s) \\ &= (d-1)^N + N + N^2\end{aligned}$$

As the dimension of the fibres is constant, $\mathcal{A}$ is flat over U by **??**. But then each homogeneous component of $\mathcal{A}$, being a direct summand, is also flat over U and its fibre dimension is the same at each point $s \in U$. In particular, whether the component in degree $N(d-2)+1$ is zero can be checked at a special point of U — and that is just what we did. Thus assertion $(*)$ holds in general and $N(d-2)$-determinacy follows.

Applying the same reasoning to the algebra

$$\mathcal{B} := \mathcal{O}_{\mathbb{C}^N \times U}/(\partial F/\partial z_1, \dots, \partial F/\partial z_N)$$

it follows that for every homogeneous polynomial with isolated singularity at the origin there is a homogeneous polynomial of degree $N(d-2)$ that is not in the Jacobi ideal of F. This fact alone suffices to prove that f is not $N(d-2)-1$ determined as we will see in a moment, **??** below. ☐

Although not essential to the proof, let us point out that a polynomial of degree $N(d-2)$ outside of $jac(f)$ can be obtained directly: If f is a homogeneous polynomial that defines an isolated singularity, its *Hessian* $H(f) = (\frac{\partial^2 f}{\partial z_i \partial z_j})$ generates the socle of $\mathbb{C}[z_1, \dots, z_N]/jac(f)$. The next result shows in particular that $f + H(f)$ is not contact equivalent to f as long as $\deg f \neq degH(f)$.

Lemma 2.2.8. *Let f and g be homogeneous polynomials in $\mathbb{C}[z_1, \dots, z_N]$. If f and g are of different positive degree, if f defines an isolated singularity and if g is not in $jac(f)$, no polynomial $f + tg$ with $t \neq 0$ is contact equivalent to f. More precisely, the Tyurina numbers, 2.2.6(1), satisfy*

$$\tau(f) > \tau(f + tg) \quad \textit{for} \quad t \neq 0 \ .$$

Proof. The last claim implies the first by 2.2.6(1). Observe that $f + tg$ and $f + t'g$ are contact equivalent if $tt' \neq 0$: Indeed, with $c = (t'/t)^{1/(k-d)}$ one has $(f + tg)(cz_1, \dots, cz_N) = c^d(f + t'g)(z_1, \dots, z_N)$. In particular, the Tyurina number is constant outside of $t = 0$ and we have to show that it jumps up at the origin. Now look at the family $F(z, t) = f(z) + tg(z)$ over $S = \mathbb{C}$ with coordinate t. The $\mathbb{C}[t]$-module $M = \mathbb{C}[z_1, \dots, z_N, t]/jac^e_S(F)$ has as its fibre at $t = s \in \mathbb{C}$ the $\mathbb{C}$-vector space $\mathbb{C}[z_1, \dots, z_N]/jac^e(f + sg)$ whose dimension equals $\tau(f + sg)$. To establish that $\tau(f + sg)$ is larger at $s = 0$ than elsewhere, it suffices now to show that M is a finite $\mathbb{C}[t]$-module on which t is a zero divisor, i.e. M is not flat at $t = 0$.

Using the Euler identity once for f and once for g, we find

$$\sum_{i=1}^{N} z_i \frac{\partial F}{\partial z_i} - dF = (df + ktg) - d(f + tg) = (k - d)tg \ .$$

By assumption, $k-d \neq 0$, whence tg is in $\operatorname{jac}^{e}_S(F)$. But g itself is not in $\operatorname{jac}^{e}_S(F)$, as it is not even in $\operatorname{jac}^{e}_S(F) + (t) = \operatorname{jac}^e(f)\mathbb{C}[z_1, \dots, z_N, t]$. Thus t is a zero divisor on $M := \mathbb{C}[z_1, \dots, z_N, t]/jac^{e}_S(F)$. On the other hand, as f has an isolated singularity, $M/tM \cong \mathbb{C}[z_1, \dots, z_N]/jac(f)$ is of finite $\mathbb{C}$-dimension and so M is a finite $\mathbb{C}[t]$-module. □

For low degrees and a small number of variables the preceding results give the following.

(a) If $f \in \mathbb{C}[X, Y, Z]$ is homogeneous of degree 3 with an isolated singularity at 0 then it is 3-determined,

(b) If $f \in \mathbb{C}[X, Y]$ is homogeneous of degree 4 it is 4-determined.

To complement the discussion, observe that in the situation of 2.2.8, a polynomial $f+tg$ with $d = \deg f = \deg g$ is contact equivalent to f iff the two polynomials are equivalent under the action of $GL(d, \mathbb{C})$ by linear coordinate changes. Thus the question of contact equivalence becomes that of the structure of orbits of this group action.

We finish this section showing that a function f is finitely determined iff it defines an isolated singularity. More precisely, the following holds.

PROPOSITION 2.2.9. *Let $f \in \mathcal{O}_{\mathbb{C}^N,0}$ be a function with $f(0) = 0$. Then the following conditions are equivalent:*

1. *$X = \{f = 0\}$ has an isolated singularity at 0.*
2. *f is finitely determined.*
3. *$\mathcal{O}_{\mathbb{C}^N,0}/\operatorname{jac}^e(f)$ is a finite dimensional $\mathbb{C}$-vectorspace, that is the Tyurina number $\tau(f)$ is finite.*

PROOF. The equivalence of (1) and (3) follows from the fact that the singular locus of X is given by the ideal $\operatorname{jac}^e(f)$. For the proof of the implication (3) $\Rightarrow$ (2) take k such that $\mathfrak{m}^k \subseteq \operatorname{jac}^e(f)$. As $\mathfrak{m}\operatorname{jac}^e(f) \subseteq (f) + \mathfrak{m}\operatorname{jac}(f)$ it follows that $\mathfrak{m}^{k+1}$ is contained in $(f) + \mathfrak{m}\operatorname{jac}(f)$, and f is $k+1$ determined by 2.2.5.

In order to prove (2) $\Rightarrow$ (1) assume that f is k-determined and defined on the neighbourhood U of 0. Applying 2.2.10 below to $U\backslash\{0\}$ with $\sigma_0 = f$ and $\sigma_j = z_j^{k+1}$, $1 \leq j \leq N$, gives that for each $(\alpha_0, \alpha_1, \dots, \alpha_N) \in \mathbb{C}^{N+1}$ outside a thin set A, the zeros of

$$F := \alpha_0 f + \sum \alpha_j z_j^{k+1}$$

form a complex submanifold of $U \setminus \{0\}$. As there exists at least one tuple outside A with $\alpha_0 \neq 0$ we get that

$$(1/\alpha_0)F = f + \sum (\alpha_j/\alpha_0) z_j^{k+1}$$

has an isolated singularity at 0. By assumption this function is contact equivalent to f and so f has an isolated singularity too. □

LEMMA 2.2.10. *Let X be a complex manifold, $\mathcal{L}$ a line bundle on X and assume that $\sigma_0, \dots, \sigma_N \in H^0(X, \mathcal{L})$are sections without common zero. For $\alpha \in \mathbb{C}^{N+1}$ set $\sigma_\alpha := \sum_{i=0}^{N} \alpha_i \sigma_i$. There is then a subset $A \subseteq \mathbb{C}^{N+1}$ of Lebesgue-measure 0 such that for $\alpha \in \mathbb{C}^{N+1} \setminus A$ the zero locus X_α of σ_α is smooth.*

Proof. Consider the set Z of zeros of $\sigma := \sum_{i=0}^{N} T_i\sigma_i$ in $\mathbb{C}^{N+1} \times X$ where $\mathbb{C}^{N+1}$ is equipped with coordinates $T_0, \ldots, T_N$. The second projection $p_2 : Z \to X$ is smooth since over $\sigma_i \neq 0$ the set Z is given by the equation

$$T_i = \sum_{j \neq i} \frac{\sigma_j}{\sigma_i} T_j \ .$$

Hence Z is smooth and applying Sard's theorem to $p_1 : Z \to \mathbb{C}^{N+1}$ gives the result. □

Remark 2.2.11. In the algebraic case the Lemma even holds with A a Zariski closed proper subset of $\mathbb{C}^{N+1}$, see [**Har**, III 10.7].

Exercise 2.2.12. (cf.[**BKn**, p.4]) (a) Generalize 2.2.7 to quasihomogeneous polynomials: A polynomial f is *quasihomogeneous* if there exist rational numbers $w_i > 0$, the *weights*, such that $f(c^{w_1}z_1, \ldots, c^{w_N}z_N) = c^d f(z_1, \ldots, z_N)$ for some d and all $c \in \mathbb{C} \setminus \{0\}$. By a theorem of K.Saito, **??**, a powerseries f is quasihomogeneous in suitable coordinates and for suitable weights iff $f \in \operatorname{jac}(f)$.

(b) Given natural numbers p, q, r greater than zero, show directly that the polynomials $f_\alpha = x^p + y^q + z^r + \alpha xyz$ are contact equivalent for all $\alpha \in \mathbb{C} \setminus \{0\}$ if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \neq 1$.

(c) With the same condition on p, q, r as in (b), show that f_0 is not contact equivalent to f_α with $\alpha \neq 0$.

(d) Discuss the remaining two cases $(p, q, r) = (3, 3, 3)$ or $(p, q, r) = (2, 4, 4)$.

Exercise 2.2.13. (The generalized Hessian) Let $f \in \mathcal{O}_{\mathbb{C}^N,0}$ be a function of multiplicity at least two so that $f \in \mathfrak{m}^2$ with $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{\mathbb{C}^N,0}$. Under these assumptions, $\operatorname{jac}(f) \subseteq \mathfrak{m}$ and consequently there is a matrix $A(f) = (a_{ij})_{1 \leq i,j \leq N}$ of functions $a_{ij} \in \mathcal{O}_{\mathbb{C}^N,0}$ such that

$$\frac{\partial f}{\partial z_i} = \sum_{j=1}^{N} a_{ij} z_j \ .$$

Show that $\det A(f)$ is well defined modulo $\operatorname{jac}(f)$ and that it generates the socle of $\mathcal{O}_{\mathbb{C}^N,0}/\operatorname{jac}(f)$ if f defines an isolated singularity at 0. What is its relation with the Hessian in case that f is (quasi-)homogeneous?

2.3. Extensions of Complex Spaces

We introduce extensions of complex spaces by coherent modules. To study such extensions, one fixes first the complex space to be extended, then the module by which one extends. Correspondingly there are various types of morphisms to be considered. After sorting out these notions, we introduce and characterize trivial extensions. As a first classification we show how to construct all extensions whose associated Jacobi map is injective. The crucial point is the interpretation of any extension of $\Omega^1_{X/\Sigma}$ by a coherent module as the Zariski-Jacobi sequence of a suitable extension. As a corollary it follows that Stein manifolds admit only trivial extensions. Finally we determine all infinitesimal automorphisms of an extension and describe locally trivial extensions through 1-cocycles of vector fields.

A *(first order) extension* of a complex space $X \in \mathbf{An}_\Sigma$ is a closed embedding $X \hookrightarrow X'$ of complex spaces over Σ such that the defining ideal $\mathcal{I} = \operatorname{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X)$

is of square zero, $\mathcal{I}^2 = 0$. In that case, $\mathcal{I}$ is naturally a coherent $\mathcal{O}_X = \mathcal{O}_{X'}/\mathcal{I}$-module. Note that the topological space underlying an extension is still X.

To study extensions, one eliminates first the usually complicated action of the $\mathcal{O}_X$-automorphisms of $\mathcal{I}$, rigidifying the notion of extension as follows.

Given a complex space X over Σ and a coherent $\mathcal{O}_X$-module $\mathcal{M}$, *an extension of* X *by* $\mathcal{M}$ *over* Σ is a pair $(i : X \hookrightarrow X', u)$ where $i : X \hookrightarrow X'$ is an extension of complex spaces over Σ and

$$u : \mathcal{M} \xrightarrow{\cong} \mathcal{I} = \operatorname{Ker}(i^* : \mathcal{O}_{X'} \to \mathcal{O}_X)$$

is a fixed isomorphism of $\mathcal{O}_X$-modules.

A *morphism of extensions* over Σ from $(i : X \hookrightarrow X', u)$, an extension of X by a coherent $\mathcal{O}_X$-module $\mathcal{M}$, to $(j : Y \hookrightarrow Y')$, an extension of Y by a coherent $\mathcal{O}_Y$-module $\mathcal{N}$, is a pair of Σ-morphisms, $f : X \to Y$ and $f' : X' \to Y'$, such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[d, "f'"', "V"] \\
Y \arrow[r, "j"] & Y'
\end{tikzcd}$$

that is, $jf = f'i$. Often the morphism $f : X \to Y$ is given and then f' is called a *lifting* of f to the given extensions.

A morphism of extensions as just defined gives rise to the following diagram of exact sequences of $\mathcal{O}_\Sigma$-modules on the topological space Y,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N} \arrow[r, "v"] \arrow[d, "\varphi"'] & \mathcal{O}_{Y'} \arrow[r, "j^*"] \arrow[d, "f'^*"'] & \mathcal{O}_Y \arrow[r] \arrow[d, "f^*"'] & 0 \\
0 \arrow[r] & f_*\mathcal{M} \arrow[r, "f_*u"] & f_*\mathcal{O}_{X'} \arrow[r, "f_*i^*"] & f_*\mathcal{O}_X &
\end{tikzcd}$$

where φ is an $\mathcal{O}_Y$-homomorphism and f'^*, f^* are $\mathcal{O}_\Sigma$-algebra morphisms. In most cases we will encounter, the morphism f will be *finite.* In that case the exact sequence on the bottom can be extended by a zero at the right, $f_*\mathcal{M}$ is a coherent $\mathcal{O}_Y$-module and the diagram constitutes just a morphism of exact sequences of coherent $\mathcal{O}_{Y'}$-modules, with the particular catch that the vertical morphisms in the middle and at the right are morphisms of $\mathcal{O}_\Sigma$-algebras.

Forgetting the actual extension, thus associating to an extension $(X \hookrightarrow X', u)$ the Σ-space X, defines a functor from the category $\mathbf{Ex}_\Sigma$ of all extensions to the category $\mathbf{An}_\Sigma$ of complex spaces over Σ,.

2.3.1. (Extensions of a fixed space) One denotes by $\mathbf{Ex}_\Sigma(X)$ the category whose objects are all extensions of a fixed space X over Σ by a coherent $\mathcal{O}_X$-module. The *morphisms of extensions of* X are those morphisms of extensions that *lift the identity on* X. Thus $\mathbf{Ex}_\Sigma(X)$ is not a full subcategory of $\mathbf{Ex}_\Sigma$ but rather the fibre over (the identity of) X in $\mathbf{An}_\Sigma$.

As all extensions of X share the same topological space X as base, a morphism of extensions of X, say from $(X \hookrightarrow X'_1, u_1)$ to $(X \hookrightarrow X'_2, u_2)$ can simply be represented by a diagram of exact sequences of $\mathcal{O}_\Sigma$-modules on X,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}_2 \arrow[r, "u_2"] \arrow[d, "\varphi"] & \mathcal{O}_{X'_2} \arrow[r] \arrow[d, "f'^*"] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M}_1 \arrow[r, "u_1"] & \mathcal{O}_{X'_1} \arrow[r] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

where $f' : X'_1 \to X'_2$ is the lift of the identity of X defining the morphism of extensions of X, and φ is the uniquely induced homomorphism of coherent $\mathcal{O}_X$-modules.

The morphism f' of extensions is an isomorphism iff $f' : X'_1 \to X'_2$ is an isomorphism of complex spaces iff f'^* is an isomorphism of $\mathcal{O}_\Sigma$-algebras iff the associated $\mathcal{O}_X$-linear map g is an isomorphism of $\mathcal{O}_X$-modules.

To abbreviate, we will simply say that X' is an extension of X, if both the embedding of X into X' and the isomorphism from the coherent $\mathcal{O}_X$-module $\mathcal{M}$ to the ideal defining X in X' are understood or not relevant to the question at hand.

Associating to an extension of X by $\mathcal{M}$ that module and to a morphism of extensions of X the induced $\mathcal{O}_X$-linear map φ defines a functor

$$\mathbf{Ex}_\Sigma(X)^{op} \to \mathbf{Coh}(X)$$

that will be investigated in detail later on, **??**.

Our first object of study will be the fibre $\mathbf{Ex}_\Sigma(X, \mathcal{M})$ of this functor over (the identity of) a given module $\mathcal{M}$.

2.3.2. (Extensions of X by $\mathcal{M}$) The category $\mathbf{Ex}_\Sigma(X, \mathcal{M})$ has as its objects *the extensions* $(X \hookrightarrow X', u)$ *of* X *over* Σ *by* $\mathcal{M}$. A morphism in this category is a morphism of extensions that induces the identity both on X and on $\mathcal{M}$. Such morphisms are necessarily isomorphisms. Extensions $(X \hookrightarrow X'_1, u_1)$ and $(X \hookrightarrow X'_2, u_2)$ of X by $\mathcal{M}$ over Σ are $\mathcal{M}$*-isomorphic* if there is a morphism between them in $\mathbf{Ex}_\Sigma(X, \mathcal{M})$. To state it explicitly once more, this means that there is a Σ-morphism $\alpha : X'_1 \to X'_2$ such that the diagrams

(∗)
$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & X'_1 \arrow[d, "\alpha"] \\
X \arrow[r] & X'_2
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{M} \arrow[r, "u_1"'] \arrow[d, equal] & \mathrm{Ker}(\mathcal{O}_{X'_1} \to \mathcal{O}_X) \arrow[d, "\alpha^*"] \\
\mathcal{M} \arrow[r, "u_2"] & \mathrm{Ker}(\mathcal{O}_{X'_2} \to \mathcal{O}_X)
\end{tikzcd}$$

commute. We will simply talk about isomorphisms of such extensions, as long as it is clear from the context that the induced automorphism on $\mathcal{M}$, and also the given map on X, is supposed to be the identity.

The set of $\mathcal{M}$-isomorphism classes of extensions of X over Σ by $\mathcal{M}$ is denoted $\mathrm{Ex}_\Sigma(X, \mathcal{M})$. Abbreviating an extension $(X \hookrightarrow X', u)$ to X', its $\mathcal{M}$-isomorphism class is $[X']$ in $\mathrm{Ex}_\Sigma(X, \mathcal{M})$. In the absolute case, where Σ is just a simple point, we write $\mathrm{Ex}(X, \mathcal{M})$.

2.3.3 (The trivial extension). To every coherent $\mathcal{O}_X$-module $\mathcal{M}$ there is associated a *trivial extension* that we now describe. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module and define on

$$\mathcal{O}_X[\mathcal{M}] := \mathcal{O}_X \times \mathcal{M}$$

the multiplication

$$(a, m)(b, n) := (ab, an + bm)$$

where a, b are local sections of $\mathcal{O}_X$ and m, n are local sections of $\mathcal{M}$. The homomorphism $a \mapsto (a, 0)$ defines $\mathcal{O}_X[\mathcal{M}]$ as $\mathcal{O}_X$-algebra. The ringed space

$$X[\mathcal{M}] := (X, \mathcal{O}_X[\mathcal{M}])$$

is isomorphic to the first infinitesimal neighbourhood of the zero section $X \hookrightarrow \mathbb{V}(\mathcal{M})$ in the linear space defined by $\mathcal{M}$ over X. In particular, $X[\mathcal{M}]$ is indeed a complex space. The first projection, $\mathcal{O}_X[\mathcal{M}] \to \mathcal{O}_X$; $(a, m) \mapsto a$, is an algebra homomorphism and together with the identity on $\mathcal{M}$ it defines $X[\mathcal{M}]$ as the *trivial extension* of X by $\mathcal{M}$.

Regarding $\mathcal{M}$ as $\mathcal{O}_X[\mathcal{M}]$-module, the second projection $\mathcal{O}_{X[\mathcal{M}]} \to \mathcal{M}$ is an $\mathcal{O}_X$-derivation.

Any extension isomorphic to $X[\mathcal{M}]$ is also called trivial and such extensions are easily characterized.

LEMMA 2.3.4. *For an extension* $(i\colon X \hookrightarrow X', u)$ *of* X *by* $\mathcal{M}$ *over* Σ *the following conditions are equivalent.*

(1) *The extension* X' *is trivial.*
(2) *There is a* Σ*-derivation* $\delta : \mathcal{O}_{X'} \to \mathcal{M}$ *such that* $\delta \circ u = id_{\mathcal{M}}$.
(3) *There is a* Σ*-morphism* $\varrho\colon X' \to X$ *retracting* i*, that is,* $\varrho \circ i = id_X$.

PROOF. If $X' \cong \mathcal{O}_X[\mathcal{M}]$ is trivial, then the $\mathcal{O}_\Sigma$-algebra isomorphism $\mathcal{O}_{X'} \xrightarrow{\cong} \mathcal{O}_{X[\mathcal{M}]}$ composed with the projection $\mathcal{O}_{X[\mathcal{M}]} \to \mathcal{M}$ yields a Σ-derivation $\delta : \mathcal{O}_{X'} \to \mathcal{M}$ that induces the identity on $\mathcal{M}$. Thus (1)$\Rightarrow$(2). For (2) $\Rightarrow$ (3), observe first that the map $id - u \circ \delta\colon \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ is an $\mathcal{O}_\Sigma$-algebra homomorphism that vanishes on $\mathcal{M}$. It induces hence an $\mathcal{O}_\Sigma$-algebra homomorphism $\varrho^*\colon \mathcal{O}_X \to \mathcal{O}_{X'}$ with $\varrho^* i^* = id_{\mathcal{O}_{X'}} - u \circ \delta$, whence a Σ-morphism $\varrho\colon X' \to X$. As $i^* \circ u = 0$, one has

$$i^* \varrho^* i^* = i^* \left(id_{\mathcal{O}_{X'}} - u \circ \delta\right) = i^* \colon \mathcal{O}_{X'} \to \mathcal{O}_X$$

and so $i^* \varrho^* = id_{\mathcal{O}_X}$. Finally assume that ϱ is a retract of i over Σ as in (3). The map $(\varrho, u) : \mathcal{O}_X \times \mathcal{M} \longrightarrow \mathcal{O}_{X'}$ yields then an isomorphism of extensions, i.e. (3) $\Rightarrow$ (1). ☐

The equivalence of (1) and (3) can be reformulated thus: For any $\mathcal{M}$, the set $\mathrm{Ex}_X(X, \mathcal{M})$ contains only the class of the trivial extension.

2.3.5 (The Zariski-Jacobi sequence of an extension). The characterization of a trivial extension through derivations in 2.3.4 (2) can be reformulated in terms of the Zariski-Jacobi sequence associated to an extension. By definition, the ideal $\mathcal{I}$ defining an extension $X \hookrightarrow X'$ over Σ by $\mathcal{M}$ satisfies $\mathcal{I} = \mathcal{I}/\mathcal{I}^2 \cong \mathcal{M}$, and thus differentiation over Σ yields the following commutative diagram

$(**)$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M} \arrow[r, "u"] \arrow[d, equal] & \mathcal{O}_{X'} \arrow[r] \arrow[d, "\bar{d}"] & \mathcal{O}_X \arrow[r] \arrow[d, "d"] & 0 \\
 & \mathcal{M} \arrow[r, "\mathbf{j}_{X'/X}"] & \Omega^1_{X'/\Sigma} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \arrow[r] & \Omega^1_{X/\Sigma} \arrow[r] & 0
\end{tikzcd}$$

where the top row is the exact sequence of $\mathcal{O}_{X'}$-modules defining the closed embedding $X \hookrightarrow X'$, the bottom row is the Zariski-Jacobi sequence of $\mathcal{O}_X$-modules associated to $X \hookrightarrow X' \to \Sigma$, and $\bar{d}$ is the composition of the universal Σ-derivation $d\colon \mathcal{O}_{X'} \to \Omega^1_{X'/\Sigma}$ followed by the projection onto $\Omega^1_{X'/\Sigma} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X$. In terms of this diagram, 2.3.4 (2) says that X' is trivial iff the Jacobi map $\mathbf{j}_{X'/X} = \bar{d} \circ u$ admits an $\mathcal{O}_X$-linear retract: If r is such a retract, so that $r\mathbf{j}_{X'/X} = id_{\mathcal{M}}$, then $\delta = r\bar{d}$ is a Σ-derivation from X' to $\mathcal{M}$ with $\delta \circ u = id_{\mathcal{M}}$, and, conversely, any such derivation δ factors uniquely through $\bar{d}$, defining a retract r.

If $\mathbf{j}_{X'/X}$ does not admit a retract, in which cases is it at least injective? The answer to this question allows it in many cases to describe *all extensions* of X by a given coherent module $\mathcal{M}$. To begin with, we show how to construct extensions starting from an $\mathcal{O}_X$-linear map onto $\Omega^1_{X/\Sigma}$.

Lemma 2.3.6. *Let $X \in \mathbf{An}_\Sigma$ be a complex space and $d : \mathcal{O}_X \to \Omega^1_{X/\Sigma}$ its universal derivation. Given a morphism of coherent $\mathcal{O}_X$-modules $\varphi\colon \mathcal{E} \to \Omega^1_{X/\Sigma}$, the fibre product*

$$\begin{tikzcd} \mathcal{O}_{X'} \arrow[r, "\varphi'"] \arrow[d, "d'"] & \mathcal{O}_X \arrow[d, "d"] \\ \mathcal{E} \arrow[r, "\varphi"] & \Omega^1_{X/\Sigma} \end{tikzcd} \tag{3}$$

defines $\mathcal{O}_{X'}$ as an $\mathcal{O}_\Sigma$-subalgebra of $\mathcal{O}_X[\mathcal{E}]$. If φ is surjective, then φ' realizes $X' = (X, \mathcal{O}_{X'})$ as an analytic extension of X by $\mathcal{M} := \ker\varphi$, furthermore $\mathcal{E} \cong \Omega^1_{X'/\Sigma} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X$ as $\mathcal{O}_X$-extensions of $\Omega^1_{X/\Sigma}$ by $\mathcal{M}$, and d' becomes identified with $\bar{d}$.

Proof. By definition $\mathcal{O}_{X'} \subseteq \mathcal{O}_X[\mathcal{E}]$. The product in $\mathcal{O}_X[\mathcal{E}]$ of local sections $(f_1, e_1), (f_2, e_2)$ from $\mathcal{O}_{X'}$ is

$$(f_1, e_1)(f_2, e_2) = (f_1 f_2, f_1 e_2 + f_2 e_1)$$

which is again a local section in $\mathcal{O}_{X'}$ as follows from the product rule

$$d(f_1 f_2) = f_1 df_2 + f_2 df_1 = f_1\varphi(e_2) + f_2\varphi(e_1) = \varphi(f_1 e_2 + f_2 e_1)\ .$$

As for trivial extensions, the projection $\varphi' : \mathcal{O}_{X'} \to \mathcal{O}_X$ is an $\mathcal{O}_\Sigma$-algebra morphism, $d'\colon \mathcal{O}_{X'} \to \mathcal{E}$ is a Σ-derivation and $\mathcal{M} = \ker\varphi$ maps isomorphically onto $\ker(\varphi')$, a square zero ideal in $\mathcal{O}_{X'}$. If φ is split surjective, then φ' admits a section and $\mathcal{O}_{X'} \cong \mathcal{O}_X[\mathcal{M}]$. Now assume that φ is just surjective. The pullback φ' is then surjective too and defines an extension of X as soon as $(X, \mathcal{O}_{X'})$ is a complex space. This is a local problem on X and we may therefore assume that X is embedded as a closed analytic subset into a space Z which is smooth over Σ. As $\Omega^1_{Z/\Sigma}$ is locally free on Z, pulling back the surjection φ along $\Omega^1_{Z/\Sigma} \to \Omega^1_{X/\Sigma}$ results locally in a split epimorphism $\tilde{\varphi}\colon \tilde{\mathcal{E}} \to \Omega^1_{Z/\Sigma}$. Repeating the preceding construction for $\tilde{\varphi}$ on Z yields locally a trivial extension $\mathcal{O}_{Z'} \cong \mathcal{O}_Z[\mathcal{M}]$ that surjects onto $\mathcal{O}_{X'}$. Thus $\mathcal{O}_{X'}$ defines locally an analytic subspace of $Z[\mathcal{M}]$.

It remains to prove that $\mathcal{E} \cong \Omega^1_{X'/\Sigma} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X$ as $\mathcal{O}_X$-extensions of $\Omega^1_{X/\Sigma}$ by $\mathcal{M}$. As $\mathcal{E}$ is an $\mathcal{O}_X$-module and the projection $d' : \mathcal{O}_{X'} \to \mathcal{E}$ is a Σ-derivation, there exists a unique $\mathcal{O}_X$-linear map $h\colon \Omega^1_{X'/\Sigma} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \to \mathcal{E}$ with $d' = h \circ \bar{d}$. The map h fits by construction into the following diagram of $\mathcal{O}_X$-modules

$$\begin{tikzcd}
& \mathcal{M} \arrow[r, "\mathbf{j}_{X'/X}"] \arrow[d, equal] & \Omega^1_{X'/\Sigma} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \arrow[r] \arrow[d, "h"'] & \Omega^1_{X/\Sigma} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{E} \arrow[r, "\varphi"] & \Omega^1_{X/\Sigma} \arrow[r] & 0
\end{tikzcd} \tag{4}$$

where the sequence on top is the Zariski-Jacobi sequence associated to the Σ-embedding $X \hookrightarrow X'$. It follows that $\mathbf{j}_{X'/X}$ is injective and then that h is bijective. □

In view of the preceding construction, an $\mathcal{O}_X$-module extension of $\Omega^1_{X/\Sigma}$ by $\mathcal{M}$ pulls back along the universal derivation $d\colon \mathcal{O}_X \to \Omega^1_{X\Sigma}$ to an extension of X over Σ by $\mathcal{M}$ and the original module extension is in turn isomorphic to the associated Zariski-Jacobi sequence. On the level of isomorphism classes this amounts to the following result.

Theorem 2.3.7. *Let X be a complex space over Σ and $\mathcal{M}$ a coherent $\mathcal{O}_X$-module. Pulling back along $d : \mathcal{O}_X \to \Omega^1_{X/\Sigma}$ defines an injection*

$$\Phi : \mathrm{Ext}^1_X\left(\Omega^1_{X/\Sigma}, \mathcal{M}\right) \hookrightarrow \mathrm{Ex}_\Sigma(X, \mathcal{M})$$

whose image consists of the classes of those extensions X' for which the associated Jacobi map $\mathbf{j}_{X'/X}$ is injective. The map Φ is bijective in the following cases:

1. *X is smooth over Σ and $\mathcal{M}$ is arbitrary, or*
2. *no local section of $\mathcal{M}$ has support in the singular locus of the structure map $X \to \Sigma$.*

Proof. Let

$$0 \to \mathcal{M} \to \mathcal{E} \to \Omega^1_{X/\Sigma} \to 0 \tag{$\mathcal{E}$}$$

be an $\mathcal{O}_X$-module extension of $\Omega^1_{X/\Sigma}$ by $\mathcal{M}$. Define the extension X' as $d^*(\mathcal{E})$, the pullback of the extension along d, so that $\mathcal{O}_{X'} := \mathcal{O}_X \times_{\Omega^1_{X/\Sigma}} \mathcal{E}$ as in Lemma 2.3.6. This defines the map

$$\begin{aligned} \Phi\colon\ \mathrm{Ext}^1_X\left(\Omega^1_{X/\Sigma}, \mathcal{M}\right) &\to \mathrm{Ex}_\Sigma(X, \mathcal{M}) \\ [\mathcal{E}] &\mapsto [(X, \mathcal{O}_{X'})] \end{aligned}$$

as X' depends up to isomorphisms of extensions only upon the class $[\mathcal{E}]$ of the given extension. As the $\mathcal{O}_X$-module extension $(\mathcal{E})$ is up to an isomorphism of such extensions the Zariski-Jacobi sequence of $X \hookrightarrow X' \to \Sigma$, the map Φ is injective and its image consists of extensions whose Jacobi map is injective. Conversely, if the Jacobi map of an extension is injective, the Zariski-Jacobi sequence constitutes an $\mathcal{O}_X$-module extension of $\Omega^1_{X/\Sigma}$ by $\mathcal{M}$ that pulls back to the given extension via d up to an $\mathcal{M}$-isomorphism of extensions. Hence the image of Φ is as claimed.

For the second part of the Theorem note that at every point $x \in X$ where $X \to \Sigma$ is smooth, the localized Zariski-Jacobi sequence

$$0 \to \mathcal{M}_x \xrightarrow{(\mathbf{j}_{X'/X})_x} \Omega^1_{X'/\Sigma} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X,x} \to \left(\Omega^1_{X/\Sigma}\right)_x \to 0$$

is (split) exact by [**Mat**, 25.2]. The kernel of $\mathbf{j}_{X'/X}$ is thus always concentrated on the singular locus of X over Σ. In the cases mentioned the Jacobi map $\mathbf{j}_{X'/X}$ is accordingly injective for every extension and the claim follows. ☐

For every complex space X over Σ and every $\mathcal{O}_X$-module $\mathcal{M}$ there are natural morphisms of groups

$$H^1(X, \Theta_{X/\Sigma} \otimes \mathcal{M}) \to H^1(X, \mathcal{D}er_\Sigma(\mathcal{O}_X, \mathcal{M})) \hookrightarrow \mathrm{Ext}^1_X(\Omega^1_{X/\Sigma}, \mathcal{M}) \tag{5}$$

each of which is an isomorphism in the smooth case. This remark, in conjunction with the preceding Theorem and Theorem B, proves the following result.

Corollary 2.3.8. *If X is a Stein manifold then every extension of X by a coherent module is trivial.* ☐

Remarks 2.3.9. (1) The second case of the Theorem above occurs in particular if $X \to \Sigma$ is generically smooth on every component of X and $\mathcal{M}$ has no $\mathcal{O}_X$-torsion. In particular, if Σ is a simple point, this is the case if X is reduced and $\mathcal{M}$ is torsionfree.

(2) The extent to which Φ fails to be surjective can be measured exactly, **??**, by means of the cotangent complex, see also 2.5.9 for a special case.

Next we describe the (infinitesimal) automorphisms of extensions. Let X', or more precisely $(X \hookrightarrow X', u)$, be some extension of X by $\mathcal{M}$ over Σ. The set $\mathrm{Aut}_{X/\Sigma}(X')$ of all *(infinitesimal) automorphisms* of the extension consists of all Σ-automorphisms $X' \to X'$ for which the diagram $(*)$ in 2.3.2 with $X'_1 = X'_2 = X'$, $u = u_1 = u_2$ commutes. Composition defines a group structure on $\mathrm{Aut}_{X/\Sigma}(X')$.

Lemma 2.3.10. *If $(i\colon X \hookrightarrow X', u)$ is an extension of X over Σ by a coherent $\mathcal{O}_X$- module $\mathcal{M}$, then there is a natural bijection*

$$\mathrm{Aut}_{X/\Sigma}(X') \xrightarrow{\cong} \mathrm{Der}_\Sigma(\mathcal{O}_X, \mathcal{M})$$

under which the composition of automorphisms is transformed into the sum of derivations.

Proof. Let $\alpha : X' \xrightarrow{\cong} X'$ be an automorphism of extensions. By definition, α induces an algebra homomorphism $\alpha^* : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ with $\alpha^* \circ u = u$ and $i^*\alpha^* = i^*$. Thus $\alpha^* - id_{\mathcal{O}_{X'}}$ factors uniquely through a map $\delta_\alpha : \mathcal{O}_X \to \mathcal{M}$ so that

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\delta_\alpha"] & \mathcal{M} \arrow[d, hook, "u"] \\
\mathcal{O}_{X'} \arrow[u, "i^*"] \arrow[r, "\alpha^* - id_{\mathcal{O}_{X'}}"] & \mathcal{O}_{X'}
\end{tikzcd}$$

commutes. It is easy to verify that δ_α is in fact a Σ-derivation. Conversely, if $\delta : \mathcal{O}_X \to \mathcal{M}$ is a Σ-derivation then

$$\alpha^* = id_{\mathcal{O}_{X'}} + u \circ \delta \circ i^* : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$$

is an algebra automorphism that defines an automorphism α in $\mathrm{Aut}_{X/\Sigma}(X')$ with $\delta_\alpha = \delta$.

If $\beta\colon X' \to X'$ is a second automorphism of extensions, then $(\alpha^* - id_{\mathcal{O}_{X'}}) \circ (\beta^* - id_{\mathcal{O}_{X'}}) = 0$ as $\beta^* - id_{\mathcal{O}_{X'}}$ takes values in $u(M)$ and $(\alpha^* - id_{\mathcal{O}_{X'}}) \circ u = 0$.

Accordingly,

$$\begin{aligned}\alpha^*\beta^* &= (id_{\mathcal{O}_{X'}} + \alpha^* - id_{\mathcal{O}_{X'}})(id_{\mathcal{O}_{X'}} + \beta^* - id_{\mathcal{O}_{X'}})\\ &= id_{\mathcal{O}_{X'}} + (\alpha^* - id_{\mathcal{O}_{X'}}) + (\beta^* - id_{\mathcal{O}_{X'}})\\ &= id_{\mathcal{O}_{X'}} + u\delta_\alpha i^* + u\delta_\beta i^*\\ &= id_{\mathcal{O}_{X'}} + u(\delta_\alpha + \delta_\beta)i^*\end{aligned}$$

and thus, by uniqueness of the corresponding derivation, $\delta_{\beta\alpha} = \delta_\alpha + \delta_\beta$. □

2.3.11. (Locally trivial extensions) In view of this result, one can describe explicitly the inclusion

$$\Psi\colon H^1(X, \mathcal{D}er_\Sigma(\mathcal{O}_X, \mathcal{M})) \hookrightarrow \mathrm{Ex}_\Sigma(X, \mathcal{M}) \tag{6}$$

obtained by composing Φ from 2.3.7 with the natural inclusion

$$H^1(X, \mathcal{D}er_\Sigma(\mathcal{O}_X, \mathcal{M})) \hookrightarrow \mathrm{Ext}^1_X(\Omega^1_{X/\Sigma}, \mathcal{M}) \ .$$

A cohomology class in $H^1(X, \mathcal{D}er_\Sigma(\mathcal{O}_X, \mathcal{M}))$ can be represented through a Čech-cocycle of $\mathcal{M}$-valued vector fields $(\delta_{ij} \in \mathrm{Der}_\Sigma(U_i \cap U_j, \mathcal{M}))_{i,j}$ with respect to an open covering $\{U_i\}$ of X. According to the preceding Lemma, these derivations define automorphisms α_{ij} of the trivial extensions $\mathcal{O}_{U_i\cap U_j}[\mathcal{M}|U_i \cap U_j]$. Gluing together the trivial extensions $\mathcal{O}_{U_i}[\mathcal{M}|U_i]$ through these automorphisms α_{ij} on the intersections $U_i \cap U_j$ yields an extension of X by $\mathcal{M}$ whose class in $\mathrm{Ex}_\Sigma(X, \mathcal{M})$ is the image under Ψ of the given cohomology class. As any extension X' that is locally trivial can be obtained by gluing trivial extensions, the image of Ψ consists precisely of the classes of locally trivial extensions. For any locally trivial extension, the Jacobi map $\mathbf{j}_{X'/X}$ is clearly injective and this reflects that Ψ factors through Φ.

2.4. The Module Structure on Extension Classes

In this section we establish the relevant functorial properties of the sets of isomorphism classes of extensions. An essential tool is Schuster's result that gives the existence of certain fibred sums in the category of complex spaces. It implies functoriality of extension classes with respect to finite morphisms. We then invoke a rather general result on functors to show that each set of isomorphism classes of extensions of X carries a natural $\Gamma(X, \mathcal{O}_X)$-module structure and that the functorial maps are linear with respect to this structure. Finally we comment upon the corresponding situation for germs of analytic spaces.

We first describe the functorial properties of $\mathrm{Ex}_\Sigma(X, \mathcal{M})$ with respect to $\mathcal{M}$.

2.4.1. If $(X \hookrightarrow X', u)$ is an extension of X by $\mathcal{M}$ over Σ and if $\varphi : \mathcal{M} \to \mathcal{N}$ is a homomorphism of coherent $\mathcal{O}_X$-modules, the map

$$(u, -\varphi)\colon \mathcal{M} \hookrightarrow \mathcal{O}_{X'} \oplus \mathcal{N} \cong \mathcal{O}_{X'[\mathcal{N}]}$$

embeds $\mathcal{M}$ as an ideal into $\mathcal{O}_{X'[\mathcal{N}]}$. The quotient $\mathcal{O}_{X'}$-algebra $\mathcal{O}_Z := (\mathcal{O}_{X'} \oplus \mathcal{N})/\mathcal{M}$ fits into the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M} \arrow[r, "u"] \arrow[d, "\varphi"] & \mathcal{O}_{X'} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{N} \arrow[r] & \mathcal{O}_Z \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \end{tikzcd}$$

of $\mathcal{O}_{X'}$-modules. By construction, the map $\mathcal{O}_{X'} \to \mathcal{O}_Z$ is a morphism of $\mathcal{O}_\Sigma$-algebras, the epimorphism $\mathcal{O}_Z \to \mathcal{O}_X$ is a morphism of such algebras and the inclusion $\mathcal{N} \to \mathcal{O}_{X'} \oplus \mathcal{N}$ provides a distinguished isomorphism onto the kernel. Hence $Z = (X, \mathcal{O}_Z)$ is an extension of X by $\mathcal{N}$ over Σ, denoted by $\varphi_* X'$.

The commutative diagram above defines not only $\varphi_* X'$ but also a morphism $f_\varphi : \varphi_* X' \to X'$ of extensions of X, 2.3.1. This morphism has the following universal property: If $(X \hookrightarrow X'', v)$ is an extension of X over Σ by $\mathcal{N}$ and if $f : X'' \to X'$ is a morphism of extensions that induces the $\mathcal{O}_X$-linear map $\varphi : \mathcal{M} \to \mathcal{N}$, then there is a unique factorization $f = \alpha \circ f_\varphi$ where $\alpha : X'' \to \varphi_* X'$ is an $\mathcal{N}$-isomorphism of extensions. As a consequence, φ_* transforms $\mathcal{M}$-isomorphisms into $\mathcal{N}$-isomorphisms and defines a natural map

$$\varphi_* \colon \operatorname{Ex}_\Sigma(X, \mathcal{M}) \longrightarrow \operatorname{Ex}_\Sigma(X, \mathcal{N}) \ .$$

Clearly $\varphi_* \left(X[\mathcal{M}]\right) \cong X[\mathcal{N}]$.

If $\varphi : \mathcal{M} \to \mathcal{N}$ and $\psi : \mathcal{N} \to \mathcal{P}$ are homomorphisms of coherent $\mathcal{O}_X$-modules, and if X' is an extension of X over Σ by $\mathcal{M}$, then $(\psi\varphi)_*(X') \cong \psi_*(\varphi_*(X'))$ as extensions of X by $\mathcal{P}$ and thus

$$(\psi\varphi)_* = \psi_*\varphi_* \colon \operatorname{Ex}_\Sigma(X, \mathcal{M}) \to \operatorname{Ex}_\Sigma(X, \mathcal{P}) \ .$$

2.4.2. With the same notations as above, assume that $(X \hookrightarrow X', u)$ is a Σ-extension of X by $\mathcal{M}$ over Σ for which the Jacobi map $\mathbf{j}_{X'/X}$ is injective. As in 2.3.7, the Zariski-Jacobi sequence of $X \hookrightarrow X' \to \Sigma$ constitutes then an $\mathcal{O}_X$-extension of $\Omega^1_{X/\Sigma}$ by $\mathcal{M}$ from which the actual extension can be reconstructed, pulling back the module extension via $d : \mathcal{O}_X \to \Omega^1_{X/\Sigma}$. As pulling back an exact sequence via a morphism on the right commutes with pushing out the exact sequence via a morphism on the left, one may construct $\varphi_* X'$ by first pushing out the Zariski-Jacobi sequence along $\varphi : \mathcal{M} \to \mathcal{N}$ and then pulling back along d. In particular, $\varphi_* X'$ is again an extension with injective Jacobi map.

If X' happens to be a locally trivial extension by M, then clearly $\varphi_* X'$ is a locally trivial extension by $\mathcal{N}$. Furthermore, if X' is glued together through a 1-Čech cocycle $(\delta_{ij})_{ij}$ of $\mathcal{M}$-valued derivations on $U_i \cap U_j$ as in 2.3.11, then $\varphi_* X'$ can be glued from trivial extensions via the 1-Čech cocycle of $\mathcal{N}$-valued derivations $(\varphi \circ \delta_{ij})_{ij}$.

These remarks just mean that for every $\mathcal{O}_X$-homomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ between coherent modules the diagram

$$\begin{tikzcd}
H^1(X, \mathcal{D}er_\Sigma(\mathcal{O}_X, \mathcal{M})) \arrow[r] \arrow[d, "H^1(X, \mathcal{D}er_\Sigma(\mathcal{O}_X, \varphi))"'] & \operatorname{Ext}^1_X(\Omega^1_{X/\Sigma}, \mathcal{M}) \arrow[r, "\Phi"] \arrow[d, "\operatorname{Ext}^1_X(\Omega^1_{X/\Sigma}, \varphi)"'] & \operatorname{Ex}_\Sigma(X, \mathcal{M}) \arrow[d, "\varphi_*"'] \\
H^1(X, \mathcal{D}er_\Sigma(\mathcal{O}_X, \mathcal{N})) \arrow[r] & \operatorname{Ext}^1_X(\Omega^1_{X/\Sigma}, \mathcal{N}) \arrow[r, "\Phi"] & \operatorname{Ex}_\Sigma(X, \mathcal{N})
\end{tikzcd}$$

commutes.

Functoriality of $\operatorname{Ex}_\Sigma(X, \mathcal{M})$ with respect to X requires that certain fibred sums exist in the category of complex spaces. Arbitrary fibred sums exist and can easily be described in the category of all *ringed spaces*. First recall the construction for topological spaces: If $f : X \to Y$ and $g : X \to X'$ are continuous maps of topological spaces, let Y' be the quotient of the disjoint union $Y \cup X'$ modulo the equivalence relation generated by $f(x) \sim g(x)$ for $x \in X$, and denote by $f' : X' \to Y'$,

$g' : Y \to Y'$ the maps induced by the inclusions. Endowed with the quotient topology Y' represents the fibred sum, meaning that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"'] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

satisfies the required universal property. To describe now the fibred sum of ringed spaces set $h = g'f = f'g : X \to Y'$.

Lemma 2.4.3. *Let*

$$\begin{tikzcd} X = (X, \mathcal{R}) \arrow[r, "f"] \arrow[d, "g"'] & Y = (Y, \mathcal{S}) \\ X' = (X', \mathcal{R}') & \end{tikzcd}$$

be a diagram of ringed spaces. The ringed space

$$\big(Y', \mathcal{S}' := f'_*(\mathcal{R}') \times_{h_*(\mathcal{R})} g'_*(\mathcal{S})\big)$$

represents the fibred sum $Y \amalg_X X'$ in the category of ringed spaces.

Proof. A commutative diagram of ringed spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "\alpha"] \\ X' \arrow[r, "\beta"] & Z = (Z, \mathcal{T}) \end{tikzcd}$$

yields a unique map of topological spaces $\gamma\colon Y' \to Z$ such that $\alpha = \gamma g'$ and $\beta = \gamma f'$. The given ring homomorphism $\mathcal{T} \to \alpha_*\mathcal{S} = \gamma_* g'_*(\mathcal{S})$ on Z defines on Y' a ring homomorphism $\gamma^{-1}(\mathcal{T}) \to g'_*(\mathcal{S})$. Analogously there is a ring homomorphism $\gamma^{-1}(\mathcal{T}) \to f'_*(\mathcal{R}')$. These morphisms fit into the commutative diagram of sheaves of rings on Y'

$$\begin{tikzcd} \gamma^{-1}(\mathcal{T}) \arrow[r] \arrow[d] & g'_*(\mathcal{S}) \arrow[d] \\ f'_*(\mathcal{R}') \arrow[r] & h_*(\mathcal{R}) \end{tikzcd}$$

that gives rise first to a unique induced morphism $\gamma^{-1}(\mathcal{T}) \to \mathcal{S}'$ and then to a unique morphism of ringed spaces $(Y', \mathcal{S}') \to (Z, \mathcal{T})$. Thus $(Y', \mathcal{S}')$ represents the fibred sum as claimed. $\square$

If f, g are morphisms of complex spaces, the fibred sum Y' in the category of ringed spaces is the fibred sum in the category of complex spaces as soon as Y' itself is a complex space. This requires suitable (finiteness) restrictions on the maps f and g. For our purposes the following result, due to Schuster[**Schu**], suffices.

Proposition 2.4.4. *If $f\colon X \to Y$ is finite and if $g : X \hookrightarrow X'$ is an extension of X by a coherent $\mathcal{O}_X$-module $\mathcal{M}$, then the ringed space $Y' := Y \amalg_X X'$ is a complex space and represents the fibred sum of f and g in the category of complex spaces. Moreover, Y' is an extension of Y by $f_*(\mathcal{M})$.*

First we consider the case that Y is an open Stein subset of $\mathbb{C}^N$.

Lemma 2.4.5. *Assume in 2.4.4 that* $Y = U \subseteq \mathbb{C}^N$ *is an open Stein subset. The fibred sum* $U' := U \amalg_X X'$ *of ringed spaces is isomorphic to the trivial extension* $U[g_*\mathcal{M}]$. *In particular,* U' *is a complex space.*

Proof. Assume f has component functions $f_1, \dots, f_N$. As f is finite, X is Stein along with Y and in the exact cohomology sequence associated to the extension X',

$$\cdots \to H^0(X', \mathcal{O}_{X'}) \to H^0(X, \mathcal{O}_X) \to H^1(X, \mathcal{M}) \to \cdots ,$$

the module $H^1(X, \mathcal{M})$ vanishes by Theorem B. Thus the functions $f_1, \dots, f_N$ can be lifted to functions on X', say $F_1, \dots, F_N$, defining a lifting $F : X' \to U$ of f. As the underlying topological spaces of U' and U are the same, the structure sheaf $\mathcal{O}_{U'}$ of the ringed space $U' = U \amalg_X X'$ equals

$$\mathcal{O}_{U'} = \mathcal{O}_U \times_{f_*(\mathcal{O}_X)} f_*(\mathcal{O}_{X'})$$

by construction. This is an extension of $\mathcal{O}_U$ by $f_*(\mathcal{M})$ as the commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & f_*\mathcal{M} \arrow[r] \arrow[d, equal] & \mathcal{O}'_U \arrow[r] \arrow[d] & \mathcal{O}_U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*\mathcal{M} \arrow[r] & f_*\mathcal{O}_{X'} \arrow[r] & f_*\mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

shows. The map F defines a morphism $F^* \colon \mathcal{O}_U \to f_*(\mathcal{O}_{X'})$ and then a morphism $(1, F^*) \colon \mathcal{O}_U \to \mathcal{O}_{U'}$. It follows that $\mathcal{O}_{U'} = \mathcal{O}_U[f_*\mathcal{M}]$ and this proves the lemma. □

Proof of the Proposition: The problem is local in Y and so we may assume that there is a closed embedding $j : Y \hookrightarrow U$, where $U \subseteq \mathbb{C}^N$ is an open Stein subset. Consider the fibred sums $U' := U \amalg_X X'$ and $Y' := Y \amalg_X X'$ as ringed spaces. We know by the previous lemma that U' is a complex space. It is therefore sufficient to show that Y' is a closed subspace of U' given by a coherent sheaf of ideals. For this observe that there is a canonical map of sheafs on U'

$$\mathcal{O}_{U'} = \mathcal{O}_U \times_{f_*(\mathcal{O}_X)} f_*(\mathcal{O}_{X'}) \longrightarrow \mathcal{O}_{Y'} = \mathcal{O}_Y \times_{f_*(\mathcal{O}_X)} f_*(\mathcal{O}_{X'})$$

which is surjective along with $\mathcal{O}_U \to \mathcal{O}_Y$ and whose kernel is $\mathcal{J} \cong \mathcal{J} \times 0$. As this ideal sheaf is coherent as a $\mathcal{O}_U$-module, and then also as a $\mathcal{O}_{U'}$-module, the result follows. □

In terms of extensions, the preceding result yields that $\mathrm{Ex}_\Sigma(X, \mathcal{M})$ is functorial with respect to finite Σ-morphisms $f \colon X \to Y$.

Corollary 2.4.6. *Let* $f : X \to Y$ *be a finite* Σ*-morphism of complex spaces and* $\mathcal{M}$ *a coherent* $\mathcal{O}_X$*-module. For every extension* $i : X \hookrightarrow X'$ *of* X *by* $\mathcal{M}$ *the fibred sum* $f_*X' := Y \amalg_X X'$ *defines an extension of* Y *by* $f_*(\mathcal{M})$. *Taking isomorphism classes defines a map*

$$f_* : \mathrm{Ex}_\Sigma(X, \mathcal{M}) \to \mathrm{Ex}_\Sigma(Y, f_*\mathcal{M}) .$$

The extension f_*X' *of* Y *is trivial iff* f *factors through* $i : X \hookrightarrow X'$.

If $g : Y \to Z$ *is another finite* Σ*-morphism, then*

$$g_*f_* = (gf)_* : \mathrm{Ex}_\Sigma(X, \mathcal{M}) \to \mathrm{Ex}_\Sigma(Z, g_*f_*\mathcal{M}) .$$

Proof. The first claim was just established. By 2.3.4, the extension $i' : Y \hookrightarrow f_*X'$ of Y is trivial iff i' admits a retract. In view of the universal property of fibred sums this is equivalent to the existence of a factorization of f over i. Associativity of forming fibred sums yields the final claim, as

$$g_*(f_*(X')) = Z \amalg_Y (Y \amalg_X X') \cong Z \amalg_X X' = (gf)_*(X')$$

□

Remarks 2.4.7. (1) The reader may easily verify that for an $\mathcal{O}_X$-linear map $\varphi : \mathcal{M} \to \mathcal{N}$ of coherent $\mathcal{O}_X$-modules the diagram

$$\begin{tikzcd}
\mathrm{Ex}_\Sigma(X,\mathcal{M}) \arrow[r, "f_*"] \arrow[d, "\varphi_*"'] & \mathrm{Ex}_\Sigma(Y, f_*\mathcal{M}) \arrow[d, "(f_*(\varphi))_*"] \\
\mathrm{Ex}_\Sigma(X,\mathcal{N}) \arrow[r, "f_*"] & \mathrm{Ex}_\Sigma(Y, f_*\mathcal{N})
\end{tikzcd}$$

commutes.

(2) Assume given a finite map $f\colon X \to Y$ of complex spaces over Σ, a coherent $\mathcal{O}_X$-module $\mathcal{M}$, and a coherent $\mathcal{O}_Y$-module $\mathcal{N}$. An $\mathcal{O}_Y$-linear map $\psi : f_*\mathcal{M} \to \mathcal{N}$ induces a map

$$(f,\psi)_* : \mathrm{Ex}_\Sigma(X,\mathcal{M}) \to \mathrm{Ex}_\Sigma(Y,\mathcal{N})$$

by composing the natural maps

$$\mathrm{Ex}_\Sigma(X,\mathcal{M}) \xrightarrow{\quad f_* \quad} \mathrm{Ex}_\Sigma(Y, f_*\mathcal{M}) \xrightarrow{\quad \psi_* \quad} \mathrm{Ex}_\Sigma(Y,\mathcal{N}) \ .$$

2.4.8. (General structure of morphisms of extensions) Defined through a fibred sum, f_*X' can also be characterized by a universal property. The construction yields first of all a Σ-morphism of complex spaces $in_2 : X' \to Y \amalg_X X' = f_*X'$ that lifts $f : X \to Y$. If now $f' : X' \to Y'$ is any Σ-morphism to an extension Y' of Y by a coherent $\mathcal{O}_Y$-module $\mathcal{N}$ that lifts the given finite morphism $f : X \to Y$ over Σ, it induces an $\mathcal{O}_Y$-linear map $\varphi' : \mathcal{N} \to f_*\mathcal{M}$. In view of the universal properties of f_* and φ'_* respectively, f' factors uniquely through both canonical morphisms $X' \to f_*X'$ and $\varphi'_*Y' \to Y'$, say

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \\
f_*X' \arrow[r, "\tilde{f}"] & \varphi'_*Y' \arrow[u]
\end{tikzcd}$$

and $\tilde{f}$ is a $f_*\mathcal{M}$-isomorphism of extensions of Y over Σ. Conversely, any $f_*\mathcal{M}$-isomorphism $\beta : f_*X' \to \varphi'_*Y'$ of extensions of Y over Σ gives rise to a lifting of f that induces φ'.

Taking into account the description of infinitesimal automorphisms of extensions in 2.3.10, it follows that given X' and Y' as before, there is either no lifting of f inducing φ' or else these liftings are naturally parametrized by $\mathrm{Der}_\Sigma(Y, f_*\mathcal{M})$, as they form a set on which these derivations act freely and transitively.

A particular nuisance, see 2.4.12 below, responsible for some of the heavy machinery later on, are those cases where there is no lifting of $f = id_X$ inducing a given $\mathcal{O}_X$-linear map φ'. It is precisely this phenomenon that prevents the construction of universal deformations allowing only for versal ones.

2.4.9. Let X' be an extension of X by $\mathcal{M}$ whose associated Jacobi map $\mathbf{j}_{X'/X}$ is injective. If $f : X \to Y$ is a finite Σ-morphism, then the Zariski-Jacobi sequences for $Y \hookrightarrow Y' = f_*X' \to \Sigma$ and for $X \hookrightarrow X' \to \Sigma$ are related through the commutative diagram

$$\begin{tikzcd}
 & f_*\mathcal{M} \arrow[r, "\mathbf{j}_{Y'/Y}"] \arrow[d, equal] & \Omega^1_{Y'/\Sigma} \arrow[r] \arrow[d] & \Omega^1_{Y/\Sigma} \arrow[r] \arrow[d, "df^*"'] & 0 \\
0 \arrow[r] & f_*\mathcal{M} \arrow[r, "\mathbf{j}_{X'/X}"] & f_*\Omega^1_{X'/\Sigma} \arrow[r] & f_*\Omega^1_{X/\Sigma} \arrow[r] & 0
\end{tikzcd} \tag{7}$$

and this implies first that $\mathbf{j}_{Y'/Y}$ is injective as well and secondly that the Zariski-Jacobi sequence on top is the pullback of the Zariski-sequence on the bottom by df^*. In terms of extensions of modules, f induces thus the map

$$f_* : \operatorname{Ext}^1_X(\Omega^1_{X/\Sigma}, \mathcal{M}) \to \operatorname{Ext}^1_Y(\Omega^1_{Y/\Sigma}, f_*\mathcal{M})$$

obtained by first applying f_* to an extension and then pulling back along $df^* : \Omega^1_{Y/\Sigma} \to f_*\Omega^1_{X/\Sigma}$.

Assume now that X' is a locally trivial extension. If $f : X \to Y$ is still finite and y is any point in Y, choose for each of the finitely many points $x \in f^{-1}(y)$ a neighbourhood U_x over which the extension is trivial. Shrinking these neighbourhoods we may assume that they are disjoint. Now there exists an open neighbourhood V of y in Y such that $f^{-1}(V) \subseteq U := \bigcup_{x \in f^{-1}(y)} U_x$ and then

$$f_*(X')|V = \prod_{x \in f^{-1}(y)} \mathcal{O}_Y[\mathcal{M}|U_x \cap f^{-1}(V)] = \mathcal{O}_Y[f_*\mathcal{M}|V]$$

whence f_* preserves locally trivial extensions. On the level of cocycles, this operation corresponds to the composition

$$f_* : H^1(X, \mathcal{D}er_\Sigma(\mathcal{O}_X, \mathcal{M})) \xrightarrow{\cong} H^1(Y, f_*\mathcal{D}er_\Sigma(\mathcal{O}_X, \mathcal{M})) \to H^1(Y, \mathcal{D}er_\Sigma(\mathcal{O}_Y, f_*\mathcal{M})$$

Again we can summarize these remarks as saying that

$$\begin{tikzcd}
H^1(X, \mathcal{D}_\Sigma(\mathcal{O}_X, \mathcal{M})) \arrow[r] \arrow[d, "f_*"'] & \operatorname{Ext}^1_X(\Omega^1_{X/\Sigma}, \mathcal{M}) \arrow[r, "\Phi"] \arrow[d, "f_*"'] & \operatorname{Ex}_\sigma(X, \mathcal{M}) \arrow[d, "f_*"'] \\
H^1(Y, \mathcal{D}er_\Sigma(\mathcal{O}_Y, f_*\mathcal{M}) \arrow[r] & \operatorname{Ext}^1_Y(\Omega^1_{Y/\Sigma}, f_*\mathcal{M}) \arrow[r, "\Phi"] & \operatorname{Ex}_\sigma(Y, f_*\mathcal{M})
\end{tikzcd} \tag{8}$$

constitutes a commutative diagram.

2.4.10. If $\Sigma \to \Xi$ is a morphism of complex spaces, any extension of X over Σ is clearly also an extension of X over Ξ by the same coherent module and there is an associated *forgetful map*

$$\operatorname{Ex}_\Sigma(X, \mathcal{M}) \to \operatorname{Ex}_\Xi(X, \mathcal{M})$$

for every coherent $\mathcal{O}_X$-module $\mathcal{M}$. These maps are natural with respect to finite Σ-morphisms $X \to Y$ and $\mathcal{O}_X$-linear maps $\mathcal{M} \to \mathcal{N}$.

If $f : X \to Y$ is a finite Σ-morphism, we get in particular a sequence of maps

$$\operatorname{Ex}_Y(X, \mathcal{M}) \to \operatorname{Ex}_\Sigma(X, \mathcal{M}) \xrightarrow{f_*} \operatorname{Ex}_\Sigma(Y, f_*\mathcal{M})$$

and, by 2.4.6, a class in $\operatorname{Ex}_\Sigma(X, \mathcal{M})$ maps to the trivial class in $\operatorname{Ex}_\Sigma(Y, f_*\mathcal{M})$ iff it is in the image of $\operatorname{Ex}_Y(X, \mathcal{M})$.

We ask the reader to similarly interpret the exact sequences

$$\mathrm{Ext}^1_X(\Omega^1_{X/Y},\mathcal{M}) \to \mathrm{Ext}^1_X(\Omega^1_{X/\Sigma},\mathcal{M}) \xrightarrow{f_*} \mathrm{Ext}^1_Y(\Omega^1_{Y/\Sigma}, f_*\mathcal{M}) \tag{9}$$

and

(10)

$$H^1(X,\mathcal{D}er_Y(\mathcal{O}_X,\mathcal{M})) \to H^1(X,\mathcal{D}er_\Sigma(\mathcal{O}_X,\mathcal{M})) \xrightarrow{f_*} H^1(Y,\mathcal{D}er_\Sigma(\mathcal{O}_Y,f_*\mathcal{M}))$$

for extensions whose Jacobi map is injective, resp. that are locally trivial.

Specializing now Schuster's result on the existence of fibred sums to the case where both maps are extensions, one gets the following property of the category $\mathbf{Ex}_\Sigma(X)$ of extensions of X.

Corollary 2.4.11. *Fibred sums exist in the category* $\mathbf{Ex}_\Sigma(X)$ *of extensions of* X *over* Σ *and they are transformed into fibre products of coherent modules under the natural functor* $\mathbf{Ex}_\Sigma(X)^{op} \to \mathbf{Coh}(X)$.

Proof. Let X'_0, X'_1, X'_2 be extensions of X by coherent $\mathcal{O}_X$-modules $\mathcal{M}_0,\mathcal{M}_1,\mathcal{M}_2$ respectively. Given morphisms $X'_0 \to X'_1$ and $X'_0 \to X'_2$ of extensions of X over Σ, we claim that $X' = X'_1 \amalg_{X'_0} X'_2$ represents their fibred sum in $\mathbf{Ex}_\Sigma(X)$. By 2.4.4, X' is a complex space over Σ and the Σ-embeddings $X \to X'_i$ combine to give a Σ-embedding $X \to X'$. To complete the proof, observe that the kernel of the map of sheaves on X,

$$\mathcal{O}_{X'} = \mathcal{O}_{X'_1} \times_{\mathcal{O}_{X'_0}} \mathcal{O}_{X'_2} \to \mathcal{O}_X$$

is canonically isomorphic to $\mathcal{M} := \mathcal{M}_1 \times_{\mathcal{M}_0} \mathcal{M}_2$. □

On the level of isomorphism classes, we only can ascertain that the functor of extension classes behaves well with respect to finite direct products.

Lemma 2.4.12. *The functor* $\mathcal{M} \mapsto \mathrm{Ex}_\Sigma(X,\mathcal{M})$ *is compatible with finite direct products.*

Proof. Let $\mathcal{M}_1,\mathcal{M}_2$ be coherent $\mathcal{O}_X$-modules and let $p_i : \mathcal{M} := \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_i$ be the i-th projection. The claim is that the canonical map

$$p = (p_{1*}, p_{2*}) : \mathrm{Ex}_\Sigma(X,\mathcal{M}_1\times\mathcal{M}_2) \to \mathrm{Ex}_\Sigma(X,\mathcal{M}_1)\times\mathrm{Ex}_\Sigma(X,\mathcal{M}_2)$$

is bijective. The preceding result shows that for given classes $[X'_i] \in \mathrm{Ex}_\Sigma(X,\mathcal{M}_i)$ the class of $X' := X'_1 \amalg_X X'_2$ provides a pre-image.

To prove injectivity, assume that $[X''] \in \mathrm{Ex}_\Sigma(X,\mathcal{M})$ is an element satisfying $p_{i*}([X'']) = [X'_i]$. The natural morphisms of extensions $p_{i*}X'' \to X''$ composed with the respective $\mathcal{M}_i$-isomorphism $X'_i \to p_{i*}X''$ define a unique $\mathcal{M}$-isomorphism $X' = X'_1 \amalg_X X'_2 \to X''$, whence $[X'] = [X'']$ in $\mathrm{Ex}_\Sigma(X,\mathcal{M})$. □

That the corresponding statement for fibred products is usually not true is due to the following: Assume given $\mathcal{O}_X$-linear maps $\varphi_i : \mathcal{M}_i \to \mathcal{M}_0$ for $i = 1, 2$ and isomorphism classes $[X'_i] \in \mathrm{Ex}_\Sigma(X,\mathcal{M}_i)$ with $\varphi_{i*}[X'_i] = [X_0]$. These data yield canonical morphisms $\varphi_{i*}X'_i \to X'_i$ and the existence of some $\mathcal{M}_0$-isomorphism of extensions $\varphi_{1*}X'_1 \cong \varphi_{2*}X'_2$. But there is no assurance that such an isomorphism can be lifted to either X'_i, and without that, one cannot form a fibred sum! Cases where lifting of isomorphisms is possible will be presented later in **??**.

Coming back to positive results, for any coherent $\mathcal{O}_X$-module $\mathcal{M}$, the set $\mathrm{Ex}_\Sigma(X,\mathcal{M})$ contains the class of the trivial extension $X[\mathcal{M}]$, thus these sets are never empty. This fact, together with the compatibility with direct products that was just established, guarantees the existence of a natural $\Gamma(X, \mathcal{O}_X)$-module structure on the sets $\mathrm{Ex}_\Sigma(X, \mathcal{M})$, as the following general result shows.

PROPOSITION 2.4.13. *Let $F : \mathbf{Coh}(X) \to \mathbf{Sets}$ be a set valued functor on the category of coherent $\mathcal{O}_X$-modules $\mathbf{Coh}(X)$. Call such a functor* non empty *if $F(0) \neq \emptyset$.*

1. *Assume that F is non empty and compatible with finite direct products. For every coherent $\mathcal{O}_X$-module $\mathcal{M}$ the set $F(\mathcal{M})$ carries then a natural $\Gamma(X, \mathcal{O}_X)$-module structure such that for a morphism $\mathcal{M} \to \mathcal{N}$ of coherent $\mathcal{O}_X$-modules the induced map $F(\mathcal{M}) \to F(\mathcal{N})$ is $\Gamma(X, \mathcal{O}_X)$-linear.*
2. *If $\varphi : F \to F'$ is a natural transformation of set valued functors on $\mathbf{Coh}(X)$ each of which is compatible with finite direct products and non empty, then $\varphi(\mathcal{M}) : F(\mathcal{M}) \to F'(\mathcal{M})$ is $\Gamma(X, \mathcal{O}_X)$-linear for every $\mathcal{M}$.*

PROOF. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module and let $p_1, p_2 : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be the natural projections and $\mathrm{add}_\mathcal{M} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be the addition map. Consider the diagram

$$F(\mathcal{M}) \times F(\mathcal{M}) \xrightarrow{(F(p_1),F(p_2))} F(\mathcal{M} \times \mathcal{M}) \xrightarrow{F(\mathrm{add}_\mathcal{M})} F(\mathcal{M}) \ .$$

By assumption $p = (F(p_1), F(p_2))$ is bijective, which allows it to define a natural addition map

$$\mathrm{add}_{F(\mathcal{M})} := F(\mathrm{add}_\mathcal{M}) \circ p^{-1} : F(\mathcal{M}) \times F(\mathcal{M}) \to F(\mathcal{M}) \ .$$

That $\mathrm{add}_{F(\mathcal{M})}$ is commutative follows immediately from the corresponding property of $\mathrm{add}_\mathcal{M}$. As F commutes with direct products, the diagonal map $F(0) \to F(0) \times F(0)$ is bijective, and, as $F(0)$ is nonempty, this set contains precisely one element, denoted by 0. Under the natural map $F(0) \to F(M)$ this yields a distinguished element in $F(M)$. That this element is indeed the zero element, respectively that the addition on $F(M)$ is associative, these properties can be expressed through commutativity of diagrams which in turn follows from the corresponding property of $\mathrm{add}_\mathcal{M}$, again using that F is compatible with direct products. In a similar way, the action of $\Gamma(X, \mathcal{O}_X)$ on $\mathcal{M}$ is given by a map

$$\Gamma(X, \mathcal{O}_X) \to \mathrm{Hom}_X(\mathcal{M}, \mathcal{M}) \ ,$$

that in turn defines by functoriality a map

$$\Gamma(X, \mathcal{O}_X) \to \mathrm{Hom}_{\mathbf{Sets}}\left(F(\mathcal{M}), F(\mathcal{M})\right) \ .$$

Checking again commutativity of the corresponding diagrams shows that this latter map defines a natural $\Gamma(X, \mathcal{O}_X)$-module structure on $F(\mathcal{M})$. Statement (2) follows along the same lines. □

As the functor $\mathrm{Ex}_\Sigma(X, -)$ satisfies the hypotheses of the preceding proposition we get the promised result.

COROLLARY 2.4.14. *Let X be a complex space over Σ. For every coherent $\mathcal{O}_X$-module $\mathcal{M}$ the set $\mathrm{Ex}_\Sigma(X, \mathcal{M})$ carries a natural $\Gamma(X, \mathcal{O}_X)$-module structure such that for any morphism $\mathcal{M} \to \mathcal{N}$ of coherent $\mathcal{O}_X$-modules the induced map $\mathrm{Ex}_\Sigma(X, \mathcal{M}) \to \mathrm{Ex}_\Sigma(X, \mathcal{N})$ is $\Gamma(X, \mathcal{O}_X)$-linear. The zero element in $\mathrm{Ex}_\Sigma(X, \mathcal{M})$ is the class of the trivial extension $X[\mathcal{M}]$.* □

In view of 2.4.1 and the proofs of 2.4.12, resp. of 2.4.13, the $\Gamma(X, \mathcal{O}_X)$-module structure is quite explicit. If $\mathrm{add}_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is again addition on $\mathcal{M}$ and if $h_s : \mathcal{M} \to \mathcal{M}$ denotes multiplication with a section $s \in \Gamma(X, \mathcal{O}_X)$ on $\mathcal{M}$, then

$$[X_1'] + [X_2'] = (\mathrm{add}_{\mathcal{M}})_*[X_1' \amalg_X X_2'] \tag{11}$$

$$s[X_1'] = (h_s)_*[X_1'] \tag{12}$$

The module structure allows more succinct formulations of earlier results.

COROLLARY 2.4.15. *Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module on a complex space X over Σ. The inclusions*

$$\begin{aligned} \Phi\colon\ \mathrm{Ext}^1_X(\Omega^1_{X/\Sigma}, \mathcal{M}) &\hookrightarrow \mathrm{Ex}_\Sigma(X, \mathcal{M}) && \textit{from}\ \ 2.3.7 \\ \Psi\colon\ H^1(X, \mathcal{D}er_\Sigma(\mathcal{O}_X, \mathcal{M})) &\hookrightarrow \mathrm{Ex}_\Sigma(X, \mathcal{M}) && \textit{from}\ \ 2.3.11 \end{aligned}$$

are functorial in M and thus $\Gamma(X, \mathcal{O}_X)$-linear.

For every finite Σ-morphism $f : X \to Y$ the sequence of $\Gamma(X, \mathcal{O}_X)$-modules

$$\mathrm{Ex}_Y(X, \mathcal{M}) \to \mathrm{Ex}_\Sigma(X, \mathcal{M}) \xrightarrow{f_*} \mathrm{Ex}_\Sigma(Y, f_*\mathcal{M})$$

is exact and natural in $\mathcal{M}$. □

PROOF. Indeed, 2.4.2 shows that $\Phi\colon \mathrm{Ext}^1_X(\Omega^1, -) \to \mathrm{Ex}_\Sigma(X, -)$, and analogously Ψ, each define a natural transformation between functors that are non empty and preserve direct products. The last assertion follows from 2.4.7 and the naturality of the maps in the sequence. □

Finally we comment upon extensions of germs of analytic spaces.

2.4.16. Let $k \to A$ be a homomorphism of rings. An extension $(p : A' \to A, u)$ of A over k by some A-module M is a surjection of k-algebras $p : A' \to A$ such that the ideal $I = \mathrm{Ker}\, p \subseteq A'$ is of square zero and $u : M \to I$ is a fixed isomorphism of A-modules. In complete analogy to the previous section one defines first morphisms of extensions, then morphisms of extensions of A, finally M-morphisms of extensions of A over k by M.

If A is a local ring with unique maximal ideal $\mathfrak{m}_A$, then a k-algebra A' that extends A is necessarily local as well with unique maximal ideal $\mathfrak{m}'_A = p^{-1}(\mathfrak{m}_A)$.

If A is a noetherian ring and M is a finite A-module, then A' is again noetherian.

In the cases of interest to us, $k \to A$ will be a morphism of *analytic* algebras (in the broad sense; to be defined?????) and in that case we only consider extensions of A by *analytic algebras* A', denoting by $\mathrm{Ex}_k(A, M)$ the M-isomorphism classes of such extensions of A over k by a fixed finite module M. Analytic algebras are local and we usually think of A as the local ring $\mathcal{O}_{S,0}$ of a germ $(S, 0)$ of some analytic space, the base ring k representing the local ring of a germ $(\Sigma, 0)$ — base points will indiscriminately be denoted by 0 as long as no confusion seems possible. In this geometric interpretation, all considerations in the present or preceding section carry over, replacing spaces by germs throughout.

If $(S, 0)$ is an analytic germ corresponding to the analytic algebra A, we write $(\hat{S}, 0)$ for the formal germ at 0 whose local ring is $\mathcal{O}_{\hat{S},0} := \hat{A}$, the complete local ring that is the $\mathfrak{m}_A$-adic completion of A. More generally, if R is just a complete local noetherian k-algebra, we also think of it as the local ring of some formal germ $(T, 0)$, even if that germ is just a point.

To summarize the salient points from sect. 1.3 for analytic algebras:

(1) An analytic germ $(S,0)$ over $(\Sigma,0)$ together with a finite $\mathcal{O}_{S,0}$-module M determines the set $\operatorname{Ex}_{\Sigma,0}(\mathcal{O}_{S,0},M)$ of M-isomorphism classes of $(S,0)$ over $(\Sigma,0)$ by M.

(2) An analytic germ $(S,0)$ over $(\Sigma,0)$ together with a finite $\mathcal{O}_{S,0}$-module M determines a trivial extension $\mathcal{O}_{S,0}[M]$ and trivial extensions can be characterized as in 2.3.4.

(3) For every analytic germ $(S,0)$ over $(\Sigma,0)$ and a finite $\mathcal{O}_{S,0}$-module M there is a canonical inclusion

$$\Phi : \operatorname{Ext}^1_{\mathcal{O}_{S,0}}(\Omega^1_{S/\Sigma,0},M) \hookrightarrow \operatorname{Ex}_{S,0}(\mathcal{O}_{S,0},M)$$

where $\Omega^1_{S/\Sigma,0}$ is the $\mathcal{O}_{S,0}$-module of differentials. The image of Φ is the set of classes of those extensions whose associated Jacobi map $\mathbf{j}_{S'/S,0} : M \to \Omega^1_{S'/\Sigma,0} \otimes_{\mathcal{O}_{S',0}} \mathcal{O}_{S,0}$ is injective.

(4) If $(S,0)$ is *smooth* over $(\Sigma,0)$, then every extension of $\mathcal{O}_{S,0}$ by a finite module is trivial. Using (2) above, this is indeed a reformulation of the lifting property, **??**, characterizing smooth algebras.

(5) For every extension $(S',0)$ of an analytic germ $(S,0)$ over $(\Sigma,0)$ by a finite $\mathcal{O}_{S,0}$-module M, the group of infinitesimal automorphisms is isomorphic to $\mathit{Der}_{\Sigma,0}(\mathcal{O}_{S,0},M) \cong \operatorname{Hom}_{(S,0)}(\Omega^1_{S/\Sigma,0},M)$.

2.4.17. Concerning the functoriality of extensions of germs, the crucial result of Schuster that guarantees the existence of certain fibred sums carries over, indeed the local version of 2.4.5 suffices.

Any $\mathcal{O}_{S,0}$-homomorphism $\varphi : M \to N$ of finite modules induces a natural map

$$\varphi_* : \operatorname{Ex}_{\Sigma,0}(\mathcal{O}_{S,0},M) \to \operatorname{Ex}_{\Sigma,0}(\mathcal{O}_{S,0},N)$$

and a finite homomorphism $f : (T,0) \to (S,0)$ of analytic germs defines for every finite $\mathcal{O}_{S,0}$-module M a natural map

$$f_* : \operatorname{Ex}_{\Sigma,0}(\mathcal{O}_{S,0},M) \to \operatorname{Ex}_{\Sigma,0}(\mathcal{O}_{T,0},f_*M)$$

where f_*M is the $\mathcal{O}_{T,0}$-module obtained from M by restricting scalars along the algebra morphism $f^* : \mathcal{O}_{S,0} \to \mathcal{O}_{T,0}$. The statements 2.4.6–2.4.14 carry over mutatis mutandis, and, in particular, $\operatorname{Ex}_{\Sigma,0}(\mathcal{O}_{S,0},M)$ is equipped with a natural $\mathcal{O}_{S,0}$-module structure, functorial in M.

2.4.18. If $p : X \to \Sigma$ is some complex space over Σ, and $x \in X$ some point, we may localize in that point to pass to the germ (X,x) over $(\Sigma,p(x))$. As this operation is exact, it preserves extensions and there are natural *localization homomorphisms*

$$\operatorname{Ex}_\Sigma(\mathcal{O}_X,\mathcal{M}) \to \operatorname{Ex}_{\Sigma,p(x)}(\mathcal{O}_{X,x},\mathcal{M}_x)$$

of $\Gamma(X,\mathcal{O}_X)$-modules for every point $x \in X$ and any coherent $\mathcal{O}_X$-module $\mathcal{M}$.

Reformulating Corollary 2.4.15 for germs, the map Ψ becomes obsolete, but in return its image for a complex space X can be described by passing to the germs (X,x) at each point $x \in X$. If X' is a locally trivial extension of X by a coherent $\mathcal{O}_X$-module $\mathcal{M}$ as in 2.3.11, then its localization (X',x) at any point $x \in X$ is trivial. Conversely, if X' is trivial at $x \in X$, then it is trivial on an open neighbourhood of x as any $\mathcal{M}_x$-isomorphism with the trivial extension $\mathcal{O}_{X,x}[\mathcal{M}_x]$ is represented by a corresponding isomorphism in a neighbourhood. Thus an extension X' of X is locally trivial iff its localization at each point of X is trivial. In view of 2.4.15 and 2.3.11 this may be stated as follows.

COROLLARY 2.4.19. *A coherent $\mathcal{O}_X$-module $\mathcal{M}$ on a complex space $p\colon X \to \Sigma$ over Σ gives rise to an exact sequence of $\Gamma(X, \mathcal{O}_X)$-modules*

$$0 \to H^1(X, \mathcal{D}er_\Sigma(\mathcal{O}_X, \mathcal{M})) \xrightarrow{\Psi} \mathrm{Ex}_\Sigma(X, \mathcal{M}) \xrightarrow{loc} \prod_{x \in X} \mathrm{Ex}_{\Sigma, p(x)}(\mathcal{O}_{X,x}, \mathcal{M}_x)$$

where loc denotes the product of the various localization maps. The sequence is functorial in $\mathcal{M}$. □

This result allows also for a quicker proof of the fact that the direct image of a locally trivial extension is again so, see 2.4.9: An extension is locally trivial iff it splits at each point and this last property is clearly preserved by finite maps.

2.4.20. The $\mathfrak{m}_{S'}$-adic topology on an extension $\mathcal{O}_{S',0}$ of $\mathcal{O}_{S,0}$ induces the $\mathfrak{m}_S$-adic topology on $\mathcal{O}_{S,0}$ and completion is an exact functor on finite modules. There are thus natural *completion homomorphisms* that are functorial in the finite $\mathcal{O}_{S,0}$-module M,

$$\mathrm{Ex}_{\Sigma,0}(\mathcal{O}_{S,0}, M) \to \mathrm{Ex}_{\Sigma,0}(\mathcal{O}_{S,0}, M)^{\wedge} \to \mathrm{Ex}_{\hat{\Sigma},0}(\mathcal{O}_{\hat{S},0}, \hat{M}) \ .$$

We will see in 2.5.7 below that for any analytic algebra A over k the A-modules $\mathrm{Ex}_k(A, M)$ are *finite* along with M and that $\mathrm{Ex}_{\hat{\Sigma},0}(\mathcal{O}_{\hat{S},0}, \hat{M})$ is the completion of the finite $\mathcal{O}_{S,0}$-module $\mathrm{Ex}_{\Sigma,0}(\mathcal{O}_{S,0}, M)$. Accordingly, the first map above is injective whereas the second one is bijective. As a consequence, an extension of analytic germs is trivial iff its formal completion is trivial.

Given an analytic germ $(S, 0)$ over $(\Sigma, 0)$ and a finite $\mathcal{O}_{S,0}$-module M, completion yields an isomorphism of $\mathcal{O}_{\hat{S},0}$-modules

$$\mathrm{Der}_{\Sigma,0}(\mathcal{O}_{S,0}, M)^{\wedge} \xrightarrow{\cong} \mathrm{Der}_{\hat{\Sigma},0}(\mathcal{O}_{\hat{S},0}, \hat{M})$$

and thus, by 2.3.10, an isomorphism of groups

$$\mathrm{Aut}_{(S,0)/(\Sigma,0)}((S', 0))^{\wedge} \xrightarrow{\cong} \mathrm{Aut}_{(\hat{S},0)/(\hat{\Sigma},0)}((\hat{S}', 0))$$

for any extension S' of S by M.

2.5. Extension Classes and Closed Embeddings

2.5.1. We now apply the general results on extensions to the case of a closed embedding $i : X \hookrightarrow Y$ over Σ. If $\mathcal{I}$ is the ideal in $\mathcal{O}_Y$ defining X, then the closed subscheme $X_1 := (X, \mathcal{O}_Y/\mathcal{I}^2)$ of Y is the *first infinitesimal neighbourhood of X in Y*. It represents an extension of X over Y by the conormal module $\mathcal{I}/\mathcal{I}^2 = i^*(\mathcal{I})$,

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_Y/\mathcal{I}^2 \to \mathcal{O}_X \to 0 \ . \tag{13}$$

If $X \hookrightarrow X'$ is any extension over Y by $\mathcal{M}$, then $i_*X' \cong Y[i_*\mathcal{M}]$ by 2.4.6, and the natural morphism $X' \to i_*X'$ embeds X' as a closed subspace into that trivial extension of Y. For this reason one calls $\mathrm{Ex}_Y(X, \mathcal{M})$ the module of classes of (Y)-*embedded extensions* of X by $\mathcal{M}$.

The first infinitesimal neighbourhood X_1 of X in Y induces every other extension of X over Y in a unique way. If namely $(i' : X \hookrightarrow X', u)$ is an extension of X

over Y by a coherent $\mathcal{O}_X$-module $\mathcal{M}$, and if $\tau : X' \to Y$ denotes the structure map to Y, then $\tau i' = i$ by definition and τ induces the $\mathcal{O}_Y$-linear map

$$u^{-1}(\tau^*|\mathcal{I}) : \mathcal{I} = \ker i^* \xrightarrow{\tau^*|\mathcal{I}} \ker i'^* \xrightarrow{u^{-1}} \mathcal{M} \ .$$

As $\ker i'^*$ is a square zero ideal, this map factors in turn uniquely through an $\mathcal{O}_X$-linear map $\varphi : \mathcal{I}/\mathcal{I}^2 \to \mathcal{M}$. The corresponding commutative diagram of $\mathcal{O}_Y$-modules

$$(*) \qquad \begin{tikzcd}
0 \arrow[r] & \mathcal{I}/\mathcal{I}^2 \arrow[r] \arrow[d, "\varphi"] & \mathcal{O}_Y/\mathcal{I}^2 \arrow[r] \arrow[d, "\tau^* \bmod \mathcal{I}^2"] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r, "u"] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

shows that $X' = \varphi_* X_1$. As $\operatorname{Der}_Y(\mathcal{O}_X, \mathcal{M}) = 0$ for every coherent $\mathcal{O}_X$-module $\mathcal{M}$, extensions of X over Y admit no infinitesimal automorphisms but the identity; two extensions that are $\mathcal{M}$-isomorphic are already equal. Thus we have the following result.

LEMMA 2.5.2. *For a closed embedding* $i : X \hookrightarrow Y$ *with defining ideal* $\mathcal{I}$, *let* $X \hookrightarrow X_1$ *be the first infinitesimal neighbourhood of* X *in* Y. *Every extension of* X *over* Y *by a coherent* $\mathcal{O}_X$*-module* $\mathcal{M}$ *is of the form* $\varphi_* X_1$ *for a unique* $\mathcal{O}_X$*-linear map* $\varphi : \mathcal{I}/\mathcal{I}^2 \to \mathcal{M}$. *The resulting bijection*

$$\begin{aligned} \operatorname{Hom}_X(\mathcal{I}/\mathcal{I}^2, \mathcal{M}) &\to \operatorname{Ex}_Y(X, \mathcal{M}) \\ \varphi &\mapsto [\varphi_* X_1] \end{aligned}$$

is an isomorphism of $\Gamma(X, \mathcal{O}_X)$*-modules that is functorial in* $\mathcal{M}$.

PROOF. Bijectivity has just been established. Functoriality in $\mathcal{M}$ follows from functoriality of $\varphi \mapsto \varphi_*$, see 2.4.1. By 2.4.14, the indicated map is thus $\Gamma(X, \mathcal{O}_X)$-linear. ☐

Aside from characterizing embedded extensions in terms of $\mathcal{O}_X$-linear maps from the conormal module, this result allows constructing extensions of X over Σ by first embedding X into a suitable space Y, then inducing extensions of X over Y from the first infinitesimal neighbourhood, and finally forgetting the embedding. To pursue this aspect further, we first comment upon the case of a closed embedding whose associated Jacobi map is injective.

2.5.3. Restricting to X the Zariski-Jacobi sequence for the embedding of the first infinitesimal neighbourhood $X_1 \hookrightarrow Y$ over Σ,

$$(14) \qquad \mathcal{I}^2/\mathcal{I}^4 \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_X \xrightarrow{\mathbf{j}_{X_1/Y}|X} \Omega^1_{Y/\Sigma} \otimes \mathcal{O}_X \xrightarrow{\pi} \Omega^1_{X_1/\Sigma} \otimes \mathcal{O}_X \to 0 \ ,$$

shows that the natural map π is an isomorphism as $\mathbf{j}_{X_1/Y}(\mathcal{I}^2) \subseteq d(\mathcal{I})\mathcal{I} \equiv 0 \bmod \mathcal{I}$. Therefore the Zariski-Jacobi sequences for the Σ-embeddings $X \hookrightarrow Y$, respectively $X \hookrightarrow X_1$ are isomorphic sequences of $\mathcal{O}_X$-modules.

Now assume for the moment that the Jacobi map $\mathbf{j}_{X/Y}$ of the given embedding is *injective* so that the Zariski-Jacobi sequence constitutes an $\mathcal{O}_X$-module extension

$$(15) \qquad 0 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\mathbf{j}_{X/Y}} \Omega^1_{Y/\Sigma} \otimes \mathcal{O}_X \to \Omega^1_{X/\Sigma} \to 0$$

of $\Omega^1_{X/\Sigma}$ by $\mathcal{I}/\mathcal{I}^2$. Let $\theta \in \operatorname{Ext}^1_X(\Omega^1_{X/\Sigma}, \mathcal{I}/\mathcal{I}^2)$ be the class of this $\mathcal{O}_X$-module extension. This class provides a second way to produce extensions of X over Σ

from $\mathcal{O}_X$-linear maps on $\mathcal{I}/\mathcal{I}^2$: given such a map $\varphi : \mathcal{I}/\mathcal{I}^2 \to \mathcal{M}$, we can push out θ along φ to obtain $\varphi_*\theta \in \operatorname{Ext}(\Omega^1_{X/\Sigma}, \mathcal{M})$. Pulling then back along $d : \mathcal{O}_X \to \Omega^1_{X/\Sigma}$ produces the extension $d^*\varphi_*\theta$ of X by $\mathcal{M}$ as in 2.3.6. Pulling back or pushing out extensions commute, and because the Zariski-Jacobi sequences for $X \hookrightarrow Y$ or for $X \hookrightarrow X_1$ are isomorphic, the extension $d^*\theta$ is $\mathcal{I}/\mathcal{I}^2$-isomorphic to X_1. Therefore we get a commutative square

$$\begin{tikzcd}
\operatorname{Hom}_X(\mathcal{I}/\mathcal{I}^2, \mathcal{M}) \arrow[r, "(\ )_*\theta"] \arrow[d, "{[(\ )_*X_1]=(\ )_*d^*\theta}"', "\cong"] & \operatorname{Ext}^1_X(\Omega^1_{X/\Sigma}, \mathcal{M}) \arrow[d, hook, "\Phi=d^*"] \\
\operatorname{Ex}_Y(X, \mathcal{M}) \arrow[r, "\text{forget } Y"] & \operatorname{Ex}_\Sigma(X, \mathcal{M})
\end{tikzcd}$$

of $\Gamma(X, \mathcal{O}_X)$-modules.

According to [**?**, X.125,Prop.5(b)], the upper map in this square is the connecting homomorphism obtained by applying $\operatorname{Hom}_X(-, \mathcal{M})$ to an exact sequence of $\mathcal{O}_X$-modules whose class is $-\theta$, for example the class of the Zariski-Jacobi sequence of $X \hookrightarrow Y$ over Σ, but with $\mathbf{j}_{X/Y}$ replaced by its *opposite*! The history of this sign is rather fascinating and [**SGA** $4\frac{1}{2}$, C.D.p.265,269], [**Del**, App.], [**SGA** 4, XVII.0.3] and [**?**] provide some of the major landmarks.

Now we come to the main result of this section. For *any* closed embedding $X \hookrightarrow Y$ the isomorphism from $\operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{M})$ to $\operatorname{Ex}_Y(X, \mathcal{M})$ composed with the forgetful map $\operatorname{Ex}_Y(X, \mathcal{M}) \to \operatorname{Ex}_\Sigma(X, \mathcal{M})$ defines a map

$$\begin{aligned} \delta\colon\ \operatorname{Hom}_X(\mathcal{I}/\mathcal{I}^2, \mathcal{M}) &\to \operatorname{Ex}_\Sigma(X, \mathcal{M}) \\ \varphi &\mapsto [\varphi_* X_1] \end{aligned}$$

that is functorial in $\mathcal{M}$ and thus is $\Gamma(X, \mathcal{O}_X)$-linear by 2.4.13(2). As $i : X \hookrightarrow Y$ is finite, there is also the $\Gamma(X, \mathcal{O}_X)$-linear map

$$\operatorname{Ex}_\Sigma(X, \mathcal{M}) \xrightarrow{i_*} \operatorname{Ex}_\Sigma(Y, i_*\mathcal{M}) \ ,$$

and these maps fit together into an exact sequence — regardless of the injectivity of $\mathbf{j}_{X/Y}$, but that case dictates the choice of sign.

Proposition 2.5.4. *(*The Kodaira-Spencer sequence for a closed embedding*)* *For every closed Σ-embedding $i : X \hookrightarrow Y$ the sequence*

$$\begin{aligned} &0 \to \operatorname{Hom}_X(\Omega^1_{X/\Sigma}, \mathcal{M}) \to \operatorname{Hom}_X(\Omega^1_{Y/\Sigma} \otimes \mathcal{O}_X, \mathcal{M}) \xrightarrow{-\mathbf{j}^{\mathcal{M}}} \operatorname{Hom}_X(\mathcal{I}/\mathcal{I}^2, \mathcal{M}) \\ &\xrightarrow{\delta} \operatorname{Ex}_\Sigma(X, \mathcal{M}) \xrightarrow{i_*} \operatorname{Ex}_\Sigma(Y, i_*\mathcal{M}) \end{aligned} \tag{16}$$

of $\Gamma(X, \mathcal{O}_X)$*-modules is exact where* $\mathbf{j}^{\mathcal{M}}$ *is the* $\mathcal{M}$*-dual of the Jacobi map*

$$\mathbf{j}_{X/Y} : \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{Y/\Sigma} \otimes_{\mathcal{O}_Y} \mathcal{O}_X \ .$$

Proof. The initial segment of the sequence is the $\mathcal{M}$-dual of the exact sequence

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{-\mathbf{j}_{X/Y}} \Omega^1_{Y/\Sigma} \otimes \mathcal{O}_X \to \Omega^1_{X/\Sigma} \to 0 \tag{17}$$

whence exactness holds at the first three terms of (16).

In view of the definition of δ, the last three terms of the sequence can be identified with the sequence

$$\operatorname{Ex}_Y(X, \mathcal{M}) \to \operatorname{Ex}_\Sigma(X, \mathcal{M}) \xrightarrow{i_*} \operatorname{Ex}_\Sigma(Y, i_*\mathcal{M})$$

that is exact by 2.4.15.

It thus remains to verify exactness at $\mathrm{Hom}_X(\mathcal{I}/\mathcal{I}^2, \mathcal{M})$. As $i_*\mathcal{M}.\mathcal{I} = 0$, a derivation $\mathcal{O}_Y \to i_*\mathcal{M}$ factors uniquely through a derivation $\mathcal{O}_Y/\mathcal{I}^2 \to i_*\mathcal{M}$. If $\vartheta \in \mathrm{Hom}_X(\Omega^1_{Y/\Sigma} \otimes \mathcal{O}_X, i_*\mathcal{M}) \cong \mathrm{Der}_\Sigma(\mathcal{O}_{X_1}, \mathcal{M})$ is such a derivation, then $\mathbf{j}^{\mathcal{M}}(\vartheta)$ equals the composition $\mathcal{I}/\mathcal{I}^2 \hookrightarrow \mathcal{O}_Y/\mathcal{I}^2 \xrightarrow{\vartheta} \mathcal{M}$. According to the definition of δ and the construction in 2.4.1, the class of $\delta(-\mathbf{j}^{\mathcal{M}}(\vartheta))$ is represented by the extension X' with $\mathcal{O}_{X'} = \mathcal{O}_{X_1}[\mathcal{M}]/(\mathcal{I}/\mathcal{I}^2)$ where $\mathcal{I}/\mathcal{I}^2$ is embedded as an ideal into $\mathcal{O}_{X_1}[\mathcal{M}]$ via the map

$$(in_1, \mathbf{j}^{\mathcal{M}}(\vartheta))\colon \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_{X_1}[\mathcal{M}] \cong \mathcal{O}_{X_1} \times \mathcal{M} \ .$$

Now consider the map $\nabla : \mathcal{O}_{X_1}[\mathcal{M}] \to \mathcal{M}$ with $\nabla(f, m) = m - \vartheta(f)$ for local sections f in $\mathcal{O}_{X_1}$, resp. m in $\mathcal{M}$. It is a Σ-derivation that vanishes on the image of $\mathcal{I}/\mathcal{I}^2$ and so it induces a derivation $\overline{\nabla} : \mathcal{O}_{X'} \to \mathcal{M}$ with $\overline{\nabla}|\mathcal{M} = id_{\mathcal{M}}$. Thus the extension X' is trivial by 2.3.4.

Conversely, assume that $[\varphi_* X_1] = \delta(\varphi)$ is the trivial extension class for some $\varphi \in \mathrm{Hom}_X(\mathcal{I}/\mathcal{I}^2, \mathcal{M})$. With $X' = \varphi_* X_1$, there is then a Σ-derivation $\mathcal{O}_{X'} \to \mathcal{M}$ that is the identity on $\mathcal{M}$. Composing this map with $\mathcal{O}_{X_1} \to \mathcal{O}_{X'}$ gives a Σ-derivation $\mathcal{O}_{X_1} \to \mathcal{M}$ whose restriction to $\mathcal{I}/\mathcal{I}^2$ is just φ. □

Remark 2.5.5. In terms of extensions, the map

$$-\mathbf{j}^{\mathcal{M}} : \mathrm{Der}_\Sigma(\mathcal{O}_Y, \mathcal{M}) \longrightarrow \mathrm{Hom}_X(\mathcal{I}/\mathcal{I}^2, \mathcal{M}) \cong \mathrm{Ex}_Y(X, \mathcal{M})$$

can be interpreted as follows. If $\vartheta : \mathcal{O}_Y \to \mathcal{M}$ is a Σ-derivation then $1 + \vartheta : \mathcal{O}_Y \to \mathcal{O}_Y[\mathcal{M}]$ is a morphism of $\mathcal{O}_\Sigma$-algebras. Denoting by $1 + \vartheta$ also the associated map $Y[\mathcal{M}] \to Y$, the fibre product

$$\begin{tikzcd}
X_\vartheta \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\
Y[\mathcal{M}] \arrow[r, "1+\vartheta"] & Y
\end{tikzcd}$$

defines an extension X_ϑ of X over Y by $\mathcal{M}$. In fact, $\mathcal{O}_{X_\vartheta}$ is by construction isomorphic to the quotient of $\mathcal{O}_{Y[\mathcal{M}]}$ modulo the ideal $\mathcal{I}$ embedded via $1 + \vartheta$ into $\mathcal{O}_{Y[\mathcal{M}]}$, or, equivalently, the quotient of $\mathcal{O}_{X_1[\mathcal{M}]}$ modulo the ideal $(1 + \vartheta)(\mathcal{I}/\mathcal{I}^2)$. But this means that $\mathcal{O}_{X_\vartheta}$ can also be obtained as the fibered sum in the diagram

$$\begin{tikzcd}
\mathcal{I}/\mathcal{I}^2 \arrow[r] \arrow[d, "-\mathbf{j}^{\mathcal{M}}(\vartheta)"'] & \mathcal{O}_{X_1} \arrow[d] \\
\mathcal{M} \arrow[r] & \mathcal{O}_{X_\vartheta}
\end{tikzcd}$$

and so X_ϑ is isomorphic to the extension associated to $-\mathbf{j}^{\mathcal{M}}(\vartheta)_* X_1$ as claimed.

This description shows also clearly why $\delta \circ \mathbf{j}^{\mathcal{M}} = 0$: Forgetting the embedding into $Y[i_*\mathcal{M}]$, or, equivalently, the structure map to Y, the Σ-extension X_ϑ of X is isomorphic to the trivial extension $X[\mathcal{M}]$.

To view $-\mathbf{j}^{\mathcal{M}}$ as constructing extensions of X over Y starting from an $i_*\mathcal{M}$-valued vector field ϑ on Y identifies it as the *Kodaira-Spencer map* associated to the deformation theory of the closed embedding $X \hookrightarrow Y$, as we will see in **??**. Moreover, the whole exact sequence will be interpreted as the Kodaira-Spencer sequence of that deformation theory, whence the name attached to it here.

A useful application of the Proposition above is obtained when Y is a Stein manifold.

Corollary 2.5.6. *If $X \hookrightarrow Y$ is a closed subspace of a Stein manifold Y given by the ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$, then*

$$\operatorname{Ex}_\Sigma(X,\mathcal{M}) \cong \operatorname{Coker}\left(\mathbf{j}^{\mathcal{M}}\colon \operatorname{Hom}_X(\Omega^1_{Y/\Sigma}\otimes\mathcal{O}_X,\mathcal{M}) \to \operatorname{Hom}_X(\mathcal{I}/\mathcal{I}^2,\mathcal{M})\right). \tag{18}$$

Proof. Indeed, $\operatorname{Ex}_\Sigma(Y, i_*\mathcal{M}) = 0$ by 2.3.8, and 2.5.4 gives the result. $\square$

The preceding considerations apply to analytic algebras, once again replacing spaces by germs throughout. If $(S,0)\to(\Sigma,0)$ is a morphism of analytic germs, it factors, for a suitable $n\in\mathbb{N}$, into a closed embedding $i:(S,0)\hookrightarrow(T,0) = (\mathbb{C}^n\times\Sigma,(0,0))$ followed by the second projection $pr_1:(T,0)\to(\Sigma,0)$. As $(T,0)$ is smooth over $(\Sigma,0)$, the module $\operatorname{Ex}_{\Sigma,0}(\mathcal{O}_{T,0}, i_*M)$ vanishes for every finite $\mathcal{O}_{S,0}$-module M. Thus the module of extensions of an analytic algebra admits the following presentation.

Corollary 2.5.7. *Let $k\to A$ be a morphism of analytic algebras and choose an algebra epimorphism $\pi: k\{\mathbf{x}\} = k\{x_1,\dots,x_n\}\to A$. With $I = \operatorname{Ker}\pi$ and $\mathbf{j}: I/I^2 \to \Omega^1_{k\{\mathbf{x}\}/k}\otimes A$ the associated Jacobi map, every finite A-module M gives rise to an exact sequence*

(19)

$$0\to\operatorname{Hom}_A(\Omega^1_{A/k},M)\to\operatorname{Hom}_A(\Omega^1_{k\{\mathbf{x}\}/k}\otimes A,M)\xrightarrow{-\mathbf{j}}\operatorname{Hom}_A(I/I^2,M)\xrightarrow{\delta}\operatorname{Ex}_k(A,M)\to 0$$

of A-modules. In particular $\operatorname{Ex}_k(A,M)$ is a finite A-module and passing to the completion yields an isomorphism of finite $\hat{A}$-modules

$$\operatorname{Ex}_k(A,M)\hat{}\xrightarrow{\cong}\operatorname{Ex}_{\hat{k}}(\hat{A},\hat{M}) .$$

Proof. The exact sequence is just 2.5.4(16) rewritten for analytic algebras, using that $k\{\mathbf{x}\}$ is smooth over k. As $\operatorname{Ex}_k(A,M)$ is represented as a quotient of the finite A-module $Hom_A(I/I^2,M)$ it is finite too. Tensoring the exact sequence of finite modules with the completion map $A\to\hat{A}$ results in an exact sequence that is canonically isomorphic to the corresponding exact sequence for the morphism $\hat{\pi}:\hat{k}[\![x_1,\dots,x_n]\!]\to\hat{A}$ and the finite $\hat{A}$-module $\hat{M}$. $\square$

Specializing further, we obtain the following explicit description for hypersurface germs.

Corollary 2.5.8. *Let $(X,0)\subseteq(\mathbb{C}^n,0)$ be the germ of a hypersurface defined by a function $0\neq g\in\mathcal{O}_{\mathbb{C}^n,0}$ and let M be a finite $\mathcal{O}_{X,0}$-module. With $jac(g) = (\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n})\subset\mathcal{O}_{\mathbb{C}^n,0}$ the Jacobian ideal of g, one has*

$$\operatorname{Ex}(\mathcal{O}_{X,0},\mathcal{M})\cong\frac{M}{jac(g)M}[\frac{\partial}{\partial g}] . \tag{20}$$

Proof. Introducing coordinates $z_1,\dots,z_n$ on $\mathbb{C}^n$, the Zariski-Jacobi sequence of $(X,0)\subseteq(\mathbb{C}^n,0)$ is isomorphic to

$$0\to\mathcal{O}_{X,0}[dg]\xrightarrow{\sum_{i=1}^n\frac{\partial g}{\partial z_i}dz_i}\bigoplus_{i=1}^n\mathcal{O}_{X,0}dz_i\to\Omega^1_{X,0}\to 0 , \tag{21}$$

where $[dg]\mapsto(g \bmod I^2)\in I/I^2$ identifies the conormal module I/I^2 as a free $\mathcal{O}_{X,0}$-module with canonical generator $[dg]$. Dualizing into M gives the result. $\square$

Note that it is customary to suppress the canonical generator $[dg]$, or its dual $[\partial/\partial g]$, from the notation. But it is sometimes useful to remember: for example, if g is a homogeneous polynomial of degree d and if M is a graded module, $[\partial/\partial g]$ is a helpful reminder that degrees in M have to be adjusted by $-d$ to get the correct degrees in the then graded module $\mathrm{Ex}(\mathcal{O}_{X,0},\mathcal{M})$.

Remark 2.5.9. Returning to a general closed embedding $X \hookrightarrow Y$ of complex spaces, set $\mathcal{J} := \mathrm{Im}(\mathbf{j}_{X/Y}) \subseteq \Omega^1_{Y/\Sigma} \otimes_{\mathcal{O}_Y} \mathcal{O}_X$, so that the sequence

$$0 \to \mathcal{J} \to \Omega^1_{Y/\Sigma} \otimes_{\mathcal{O}_Y} \mathcal{O}_X \to \Omega^1_{X/\Sigma} \to 0$$

is exact. Dualizing into $\mathcal{M}$ one obtains a comparison map from the associated long exact sequence of $\mathrm{Ext}(-,\mathcal{M})$'s to the exact sequence 2.5.4(16) that involves the map Φ from 2.3.7. The relevant piece of the resulting diagram with exact rows and columns,

(22)

$$\begin{tikzcd}
\cdots \arrow[r, "-\mathbf{j}^{\mathcal{M}}_{X/Y}"] & \mathrm{Hom}_X(\mathcal{I}/\mathcal{I}^2,\mathcal{M}) \arrow[r, "\delta"] & \mathrm{Ex}_\Sigma(X,\mathcal{M}) \arrow[r] & \mathrm{Ex}_\Sigma(Y,i_*\mathcal{M}) \\
\cdots \arrow[r] & \mathrm{Hom}_X(\mathcal{J},\mathcal{M}) \arrow[u, equal] \arrow[r] & \mathrm{Ext}^1_X(\Omega^1_{X/\Sigma},\mathcal{M}) \arrow[u, "\Phi_X"] \arrow[r] & \mathrm{Ext}^1_Y(\Omega^1_{Y/\Sigma},i_*\mathcal{M}) \arrow[u, "\Phi_Y"] \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

allows the following conclusions that are left as exercises:

If $\mathbf{j}_{X/Y}$ is injective, so that $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{J}$, then $\mathrm{Im}\,\delta \subseteq \mathrm{Im}\,\Phi_X$. If Φ_Y is an isomorphism, cf. 2.4.15, then Φ_X fits into an exact sequence

$$0 \to \mathrm{Ext}^1_X(\Omega^1_{X/\Sigma},\mathcal{M}) \xrightarrow{\Phi_X} \mathrm{Ex}_\Sigma(X,\mathcal{M}) \to \mathrm{Hom}_X(\mathcal{K},\mathcal{M}) \to \mathrm{Ext}^1_X(\mathcal{J},\mathcal{M})$$

where $\mathcal{K} := \mathrm{Ker}\,\mathbf{j}_{X/Y}$ is the kernel of the Jacobi map. If $\mathrm{Ex}_\Sigma(Y,i_*\mathcal{M}) = 0$, then the last term in this sequence can be identified with $\mathrm{Ext}^2_X(\Omega^1_{X/\Sigma},\mathcal{M})$.

2.5.10. (Embeddings into Projective Space) If a complex space is embeddable into a Stein manifold, the modules of extensions can be calculated using 2.5.6 above. If $i : X \subset \mathbb{P}^n$ is a closed embedding into a complex projective space, a new phenomenon occurs: the first Chern class of the embedding line bundle $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^n}(1)$ creates an obstruction.

For simplicity, we restrict ourselves to the absolute case where Σ is just a simple point, suppressed as usual from the notation. If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, then 2.3.7, 2.3.9 give isomorphisms

$$H^1(\mathbb{P}^n,\Theta_{\mathbb{P}^n} \otimes i_*\mathcal{M}) \xrightarrow{\cong} \mathrm{Ext}^1_{\mathbb{P}^n}(\Omega^1_{\mathbb{P}^n},i_*\mathcal{M}) \xrightarrow{\cong} \mathrm{Ex}(\mathbb{P}^n,\imath_*\mathcal{M}) \ .$$

Identifying $\mathbb{P}^n = \mathbb{P}_{\mathbb{C}}(V)$ as the projective space of hyperplanes in the $(n+1)$-dimensional complex vector space V, let

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1) \otimes_{\mathbb{C}} V \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

be the *Euler sequence* on $\mathbb{P}^n$, see [**Har**, II.Thm.8.13].

Taking global sections, the connecting homomorphism $H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}) \to H^1(\mathbb{P}^n,\Omega^1_{\mathbb{P}^n})$ maps the constant function 1 to the first Chern class $c = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. The canonical isomorphism $H^1(\mathbb{P}^n,\Omega^1_{\mathbb{P}^n}) \cong \mathit{Ext}^1(\mathcal{O}_{\mathbb{P}^n},\Omega^1_{\mathbb{P}^n})$

identifies then the first Chern class with the class $[e]$ of the extension given by the Euler sequence, see once again [**Bou**, X.126,Cor.1(a)].

Now apply $\mathrm{Hom}_{\mathbb{P}^n}(-, i_*\mathcal{M})$ to the Euler sequence to obtain the relevant piece of the associated long exact sequence

$$\mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(-1)\otimes_{\mathbb{C}}V, \imath_*\mathcal{M}) \to \mathrm{Ext}^1_{\mathbb{P}^n}(\Omega^1_{\mathbb{P}^n}, i_*\mathcal{M}) \xrightarrow{\gamma} \mathrm{Ext}^2_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, i_*\mathcal{M}) \to \mathrm{Ext}^2_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(-1)\otimes_{\mathbb{C}}V, \imath_*\mathcal{M})$$

in which the connecting homomorphism γ maps a class ξ to $\xi \cdot [e]$, the product being the usual Yoneda or cup product.

Denoting by $V^\vee$ the dual vector space of V, and rewriting the terms as cohomology groups

$$(**)\quad \mathrm{Ext}^i_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(-1)\otimes_{\mathbb{C}}V, \imath_*\mathcal{M}) \cong H^i(\mathbb{P}^n, i_*\mathcal{M}(1))\otimes_{\mathbb{C}}V^\vee \cong H^i(X, \mathcal{M}\otimes\mathcal{L})\otimes_{\mathbb{C}}V^\vee$$

it follows that the outer terms in the exact sequence will vanish as soon as $\mathcal{L}$ is sufficiently ample with respect to $\mathcal{M}$. In that case, the module of extensions of $\mathbb{P}^n$ by $i_*\mathcal{M}$ becomes identified with $H^2(\mathcal{O}_X, \mathcal{M})$ through the map corresponding to γ, that is through the cup product with the first Chern class c.

In summary we have thus the following result.

PROPOSITION 2.5.11. *Let $i : X \subset \mathbb{P}^n$ be a closed embedding of a complex space X with defining ideal $\mathcal{I}$, and let $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^n}(1)$ denote the embedding line bundle. If a coherent $\mathcal{O}_X$-module $\mathcal{M}$ satisfies $H^j(X, \mathcal{M}\otimes\mathcal{L}) = 0$ for $j = 1, 2$, then the Kodaira-Spencer sequence for the given embedding and module takes the form*

$$\cdots \xrightarrow{-\mathbf{j}^{\mathcal{M}}_{X/\mathbb{P}^n}} \mathrm{Hom}_X(\mathcal{I}/\mathcal{I}^2, \mathcal{M}) \xrightarrow{\delta} \mathrm{Ex}(X, \mathcal{M}) \xrightarrow{i_*(\ )\cup c} H^2(\mathcal{O}_X, \mathcal{M})$$

where c is the first Chern class of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. □

At this stage we have no information about how obstructive the first Chern class will be for general X as we can not yet continue the Kodaira-Spencer sequence to the right. But if the Jacobi map of the embedding $X \hookrightarrow \mathbb{P}^n$ is injective, we may instead look at the long exact sequence obtained from applying $\mathrm{Hom}_X(-, \mathcal{M})$ to the Zariski-Jacobi sequence, see 2.5.9 above. The classical example where the obstruction occurred for the first time is a $K3$-surface embedded as a quartic surface in $\mathbb{P}^3$:

EXAMPLE 2.5.12. Let $i : X \hookrightarrow \mathbb{P}^3$ be a reduced surface defined by a quartic $F \in \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$. The conormal module is then $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_X(-4)$ and the Jacobi map $\mathbf{j}_{X/\mathbb{P}^3}$ is injective. Furthermore, the canonical module is trivial, $\omega_X \cong \mathcal{O}_X$, and Grothendieck-Serre duality, [**Har**], yields thus $H^2(X, \mathcal{M}) \cong \mathrm{Hom}_X(\mathcal{M}, \mathcal{O}_X)^\vee$ for every coherent $\mathcal{O}_X$-module $\mathcal{M}$.

Note further that $\mathrm{Ext}^j(\mathcal{I}/\mathcal{I}^2, \mathcal{M}) \cong H^j(X, \mathcal{M}(4))$ for each integer j. Finally, identifying $\mathrm{Ex}(X, \mathcal{M}) \cong \mathrm{Ext}^1_X(\Omega^1_X, \mathcal{M})$ and applying $\mathrm{Hom}_X(-, \mathcal{M})$ to the Zariski-Jacobi sequence, the module of classes of extensions of X by $\mathcal{M}$ appears in an exact sequence

$$\cdots \to H^0(X, \mathcal{M}(4)) \xrightarrow{\delta} \mathrm{Ex}(X, \mathcal{M}) \xrightarrow{(\ )\cup c} \mathrm{Hom}_X(\mathcal{M}, \mathcal{O}_X)^\vee \to 0$$

as soon as $H^j(X, \mathcal{M}(1)) = 0$ for $j = 1, 2$. These conditions are satisfied for $\mathcal{M} = \mathcal{O}_X$; and as $\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X) = \mathbb{C}$, the first Chern class defines a surjection from $\mathrm{Ex}(X, \mathcal{O}_X)$ onto $\mathbb{C}$, obstructing the embeddability of an extension into $\mathbb{P}^3$.

The initial segment of the Kodaira-Spencer sequence in this case can be understood as follows. $\mathrm{Hom}_X(\Omega^1_X, \mathcal{O}_X) = \mathfrak{a}ut(X)$ is the tangent Lie algebra of the automorphism group of X, whereas

$$\mathrm{Hom}_X(\Omega^1_{\mathbb{P}^3} \otimes \mathcal{O}_X, \mathcal{O}_X) \cong \mathrm{Hom}_{\mathbb{P}^3}(\Omega^1_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) = \mathfrak{p}gl(4\mathbb{C})$$

represents the tangent Lie algebra of all automorphisms of $\mathbb{P}^3$. The Kodaira-Spencer sequence becomes accordingly

$$0 \to \mathfrak{p}gl(4, \mathbb{C})/\mathfrak{a}ut(X) \to H^0(X, \mathcal{O}_X(4)) \to \mathrm{Ex}(X, \mathcal{O}_X) \to \mathbb{C} \to 0\ .$$

Note that $\dim_{\mathbb{C}}\mathfrak{p}gl(4, \mathbb{C}) = 15$, and that $H^0(X, \mathcal{O}_X(4))$, the vector space of quartic polynomials in four variables modulo the defining quartic F, is of dimension 34. Accordingly we find

$$\begin{aligned}\dim_{\mathbb{C}} \mathrm{Ex}(X, \mathcal{O}_X) &= 1 + \dim_{\mathbb{C}}H^0(X, \mathcal{O}_X(4) - \dim_{\mathbb{C}}\mathfrak{p}gl(4, \mathbb{C}) + \dim_{\mathbb{C}}\mathfrak{a}ut(X)\\ &= 20 + \dim_{\mathbb{C}}\mathfrak{a}ut(X)\end{aligned}$$

and $\dim_{\mathbb{C}}\mathfrak{a}ut(X)$ is easily calculated: The Euler sequence for Ω^1_X shows that it equals the dimension of linear relations among the partial derivatives $\frac{\partial F}{\partial z_i}$, $i = 0, \ldots, 3$. If X is smooth, those partial derivatives form a regular sequence and the first nonzero relation occurs in degree 3, thus we obtain the classical result that a smooth quartic in $\mathbb{P}^3$ admits precisely a 20-dimensional vector space of extensions by $\mathcal{O}_X$.

Taking on the other hand a union of four planes in general position, so that $F = z_0z_1z_2z_3$ in corresponding homogeneous coordinates, then $\sum_{i=0}^3 \alpha_i z_i \frac{\partial F}{\partial z_i} = 0$ iff $\sum_{i=0}^3 \alpha_i = 0$. These are easily seen to be all linear relations and accordingly there is a 23-dimensional vectorspace of extensions by the structure sheaf.

Exercise 2.5.13. Repeat the preceding discussion for a reduced hypersurface X of degree d in $\mathbb{P}^n$ with $n \geq 3$.

1. Show that $\mathrm{Ext}^1(\Omega^1_{\mathbb{P}^n}, \mathcal{O}_X) = 0$ unless $d = 4, n = 3$ and conclude that the quartic surfaces are the only hypersurfaces for which the first Chern class yields an effective obstruction for extensions by the structure sheaf.
2. Conclude that except for $d = 4, n = 3$ one has
$$\dim_{\mathbb{C}} \mathrm{Ex}(X, \mathcal{O}_X) = \binom{n+d}{d} - (n+1)^2 + \dim_{\mathbb{C}}\mathfrak{a}ut(X)$$
and that $\mathfrak{a}ut(X) = 0$ for a smooth hypersurface of degree $d \geq 3$.
3. For a nonsingular quadric, $\mathfrak{a}ut(X) \cong \mathfrak{o}(n+1, \mathbb{C})$ has dimension $(n+1)n/2$ and thus $\dim_{\mathbb{C}} \mathrm{Ex}(X, \mathcal{O}_X) = 0$.

Finally consider the case of complex curves.

Corollary 2.5.14. *Let C be a compact Riemann surface and $\mathcal{M}$ a coherent $\mathcal{O}_C$-module $\mathcal{M}$. If $\mathcal{L}$ is a very ample line bundle on C such that $H^1(C, \mathcal{M} \otimes \mathcal{L}) = 0$, then every extension of C by $\mathcal{M}$ can be induced from the first infinitesimal neighbourhood of C in the projective embedding given by the complete linear series $\mathbb{P}(H^0(C, \mathcal{L}))$.*

In particular, every extension of C by a coherent module can be embedded into some projective space.

PROOF. As $H^i(C,-)$ vanishes on any coherent $\mathcal{O}_C$-module for $i > 1$, the preceding proposition shows that the condition $H^1(C, \mathcal{M} \otimes \mathcal{L}) = 0$ alone guarantees already surjectivity of the map $\delta : \mathrm{Hom}_C(\mathcal{I}/\mathcal{I}^2, \mathcal{M}) \to \mathrm{Ex}(C, \mathcal{M})$ where $\mathcal{I}/\mathcal{I}^2$ is the conormal bundle of the projective embedding provided by $\mathcal{L}$. The last assertion follows as for every coherent module $\mathcal{M}$ there are plenty of very ample line bundles such that the vanishing condition is satisfied. $\square$

EXERCISE 2.5.15. The situation becomes particularly simple for the projective line, $C = \mathbb{P}^1$. A coherent $\mathcal{O}_{\mathbb{P}^1}$-module decomposes into a direct sum of its torsion submodule and copies of line bundles $\mathcal{O}_{\mathbb{P}^1}(a)$ for various $a \in \mathbb{Z}$. Due to additivity, it suffices to understand the extensions by those direct summands. Show that $\mathrm{Ex}(\mathbb{P}^1, \mathcal{M}) = 0$ for a torsion sheaf and that

$$\dim_{\mathbb{C}} \mathrm{Ex}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) = \max(0, -a-3) \ .$$

For $a \leq -4$, an extension of $\mathbb{P}^1$ by $\mathcal{O}_{\mathbb{P}^1}(a)$ can be embedded into $\mathbb{P}^n$ as soon as $n \geq -a-2$. The extension is then induced from the first infinitesimal neighbourhood of the rational normal curve $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}(H^0(\mathbb{P}1, \mathcal{O}_{\mathbb{P}^1}(n)))$.

CHAPTER 3

Formal deformation theories

3.1. Fibrations in Groupoids and Deformation Theories

In this section we will introduce the basic notion of a deformation theory. For later purposes it is convenient to do this in a quite general setting. We will illustrate these notions with the examples of deformations of complex spaces and deformations of coherent modules. It is convenient to use the abstract language of fibrations in categories and fibrations in groupoids which we will describe first.

Definition 3.1.1. A fibration in categories is a functor

$$p : \mathbf{F} \to \mathbf{C}$$

with the following properties:

FC1: For every morphism $f : S' \to S$ in $\mathbf{C}$ and every object a in $\mathbf{F}$ over S, i.e. $p(a) = S$, there is a morphism $\tilde{f} : a' \to a$ over f which is cartesian, i.e. $\tilde{f}$ satisfies the following universal property: For every morphism $g : b \to a$ over f there is a unique morphism $g' : b \to a'$ over $\mathrm{id}_{S'}$ with $\tilde{f}g' = g$. In other words, every diagram with solid arrows

$$\begin{tikzcd}
b \arrow[dr, "g"] \arrow[dd, dotted, "g'"'] & & & S' \arrow[dr, "f"] \arrow[dd, equal] & \\
& a & \text{over} & & S \\
a' \arrow[ur, "\tilde{f}"'] & & & S' \arrow[ur, "f"'] &
\end{tikzcd}$$

can be completed as indicated by the dotted arrow.

(FC2): Compositions of cartesian morphisms are cartesian.

The category $\mathbf{C}$ will be often called the *basis* of the fibration. In the following we will denote the objects of $\mathbf{C}$ by capital letters whereas the objects of $\mathbf{F}$ are written in lower case. If a is an object of $\mathbf{F}$ over S, i.e. $p(a) = S$ then we also often write simply $a \mapsto S$ (although this is *not* a morphism). If the morphism $\tilde{f} : a' \to a$ over $f : S' \to S$ is cartesian then the object a' is often denoted by $a' = a \times_S S'$ or also sometimes by $f^*(a)$. Since composition of cartesian morphisms are cartesian, we have a canonical isomorphism

$$a \times_S S'' \cong (a \times_S S') \times_{S'} S'',$$

if $S'' \to S'$ is further morphism in C.

The reader may easily verify that the axioms (FC1), (FC2) above are equivalent to the following property:

(FC): Let $f : S' \to S$ be a morphism in C and $a \in F$ an object over S. Then there is a morphism $\tilde{f} : a' \to a$ over f such that every diagram of solid arrows

$$\begin{tikzcd}
b \arrow[rr, mapsto] \arrow[dr] \arrow[dd, dotted] & & p(b) \arrow[dr] \arrow[dd] & \\
& a \arrow[rr, mapsto] & & S \\
a' \arrow[ur, "\wr"'] \arrow[rr, mapsto] & & S' \arrow[ur, "\wr"'] &
\end{tikzcd}$$

can be completed with a unique morphism $b \to a'$ as indicated by the dotted arrow.

A standard example of fibration in groupoids is given by set valued functors.

Example 3.1.2. Let $F : \mathbf{C}^0 \to \mathbf{Sets}$ be a functor. Then we can associate to F a fibration $p : \mathbf{F} \to \mathbf{C}$ in the following way. The objects of $\mathbf{F}$ are pairs (S, a) with $a \in F(S)$, and a morphism $(S, a) \to (T, b)$ consists in a morphism $f : S \to T$ with $F(f)(b) = a$. Then obviously every morphism in $\mathbf{F}$ is cartesian, and the fibers $\mathbf{F}(S)$ are just the sets $F(S)$ considered as a discrete category, i.e. the only morphisms are the identities.

Conversely, given a fibration $p : \mathbf{F} \to \mathbf{C}$ in categories there is an associated functor of isomorphism classes

$$[\mathbf{F}] : \mathbf{C}^0 \longrightarrow \mathbf{Sets},$$

where $[\mathbf{F}(S)]$ is the set of isomorphism classes of $\mathbf{F}(S)$. In this way fibrations in categories and set valued functors are closely related. However, fibrations in groupoids carry much more information, and they arise in a much more natural way. Especially in deformation theory it will turn out that keeping track of the automorphism of deformations will be very useful.

3.1.3. For a fibration of categories $p : \mathbf{F} \to \mathbf{C}$ one can form the *fibers* over an object S in $\mathbf{C}$ by considering all objects a in $\mathbf{F}$ over S and all morphisms in $\mathbf{F}$ over id_S. These fibers will be denoted by $\mathbf{F}(S)$. Every morphism $f : S' \to S$ induces a so called *inverse image functor* $f^*\mathbf{F}(S) \to \mathbf{F}(S')$ by $f^*(a) := a \times_S S'$. Clearly, if $g : S'' \to S'$ is a further morphism then $(fg)^* = g^* f^*$.

In the following it is also convenient to have the dual notion of cofibration:

Definition 3.1.4. A functor $p : \mathbf{F} \to \mathbf{C}$ will be called a cofibration if the functor of the opposite categories $p^0 : \mathbf{F}^0 \to \mathbf{C}^0$ is a fibration.

Similarly as above, one can form the fibres $\mathbf{F}(S)$. In this case a morphism $f : S' \to S$ induces dually a so called *direct image functor* $f_* : \mathbf{F}(S') \to \mathbf{F}(S)$.

There are many natural examples of such fibred categories in complex analysis.

Examples 3.1.5. (1) Let Σ be a fixed complex space and let $\mathbf{Mod}_\Sigma$ be the category of all pair $(S, \mathcal{M})$, where $S \to \Sigma$ is a complex space over Σ and $\mathcal{M}$ is an $\mathcal{O}_S$-module. A morphism

$$(S, \mathcal{M}) \longrightarrow (T, \mathcal{N})$$

in $\mathbf{Mod}_\Sigma$ consists of a Σ-morphism $f : S \to T$ and a $\mathcal{O}_S$-linear map $\varphi : f^*(\mathcal{N}) \to \mathcal{M}$. Then the functor

$$p : \mathbf{Mod}_\Sigma \longrightarrow \mathbf{An}_\Sigma \quad \text{with} \quad (S, \mathcal{M}) \longmapsto S$$

is a fibration. For a morphism $f : S \to T$ and an $\mathcal{O}_T$-module $\mathcal{N}$ the usual pullback $f^*(\mathcal{N})$ defines a cartesian morphism

$$(S, f^*(\mathcal{N})) \to (T, \mathcal{N}) \ .$$

The fiber of the functor p over a complex space S is just $\mathbf{Mod}(S)$, the category of $\mathcal{O}_S$-modules.

(2) Considering the full subcategory $\mathbf{Coh}_\Sigma$ all $(S, \mathcal{M})$ in $\mathbf{Mod}_\Sigma$, for which $\mathcal{M}$ is a coherent $\mathcal{O}_S$-module, gives a fibred category $\mathbf{Coh}_\Sigma \to \mathbf{An}_\Sigma$.

(3) Replacing sheaves of modules by sheaves of $\mathcal{O}_S$-algebras (and of course the morphisms $\varphi : f^*\mathcal{N} \to \mathcal{M}$ above by morphism of $\mathcal{O}_S$-algebras) we get a fibred category $\mathbf{Alg}_\Sigma \to \mathbf{An}_\Sigma$.

The most important examples for us are those arising in deformation theory.

Examples 3.1.6. (1) (Deformations of complex spaces). For a complex space $S \in \mathbf{An}_\Sigma$ consider as objects of $\mathbf{F}$ over S all flat morphisms $X \to S$. If $a = (X \to S)$, $a' = (X' \to S')$ are objects of $\mathbf{F}$ then a morphism $a' \to a$ is a cartesian diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[d] \\
S' \arrow[r] & S \ .
\end{tikzcd}$$

Observe that for a as above and $f : S' \to S$ there is always a pullback $a \times_S S'$ given by the usual fibre product $X' = X \times_S S'$, which is again flat over S'.

(2) (Deformations of modules). Let $X \to \Sigma$ be a fixed morphism of complex spaces. For a complex space $S \in \mathbf{An}_\Sigma$ let $\mathbf{F}(S)$ be the category of coherent $\mathcal{O}_{X \times_\Sigma S}$-modules $\mathcal{M}$ which are flat over S. The morphism $(T, \mathcal{N}) \to (S, \mathcal{M})$ are again pairs of morphism $f : T \to S$, $\varphi : (1 \times_\Sigma f)^* (\mathcal{M}) \to \mathcal{N}$, where f is a Σ-morphism and φ is an isomorphism.

A basic difference between the examples 3.1.5 and 3.1.6 is that for the latter ones the fibers $\mathbf{F}(S)$ of the fibrations $\mathbf{F} \to \mathbf{An}_\Sigma$ have only *isomorphisms* as morphism.

3.1.7. To investigate such fibrations recall first that a groupoid is a category in which all morphisms are isomorphisms. A typical example is given by a G-set X where G is a group acting on X: The objects are the elements of X, and the morphism $x_1 \to x_2$ are the elements $g \in G$ transporting x_1 to x_2, i.e. $g \cdot x_1 = x_2$. In this case the set of isomorphism classes is the set of orbits X/G, and the set of automorphism of $x \in X$ is the stabilizes subgroup G_x. In general, up to an equivalence of categories the structure of a groupoid $\mathbf{G}$ is determined by its set of isomorphism classes $[\mathbf{G}]$ and by the family of groups $G_{\bar{a}}$, $\bar{a} \in [\mathbf{G}]$, where $G_{\bar{a}} \cong \mathrm{Aut}(a)$ if $\bar{a}$ is represented by $a \in \mathbf{G}$. Observe that for isomorphic objects $a, b \in \mathbf{G}$ the groups $\mathrm{Aut}(a)$ and $\mathrm{Aut}(b)$ are isomorphic.

With this terminology, the fibers $\mathbf{F}(S)$ in the examples 3.1.6 are groupoids. Therefore it is convenient to introduce the following notation.

Definition 3.1.8. A (co-)fibration of categories $p : \mathbf{F} \to \mathbf{C}$ is called a *(co-)fibration in groupoids* if the fibers $\mathbf{F}(S)$ are all groupoids.

An equivalent characterization of such (co-)fibrations is that a morphism $f : a \to b$ in $\mathbf{F}$ is an isomorphism iff $p(f)$ is an isomorphism.

As mentioned above the examples treated in 3.1.6 are fibrations in groupoids whereas e.g. the fibred category of modules $\mathbf{Mod}_\Sigma \to \mathbf{An}_\Sigma$, see 3.1.5, is not fibred in groupoids (since there are many homomorphism of modules over a fixed complex space S which are not isomorphisms).

REMARK 3.1.9. In a fibration in groupoids $p : \mathbf{F} \to \mathbf{C}$, every morphism $a \to b$ in $\mathbf{F}$ is cartesian, i.e. $a \cong b \times_{p(b)} p(a)$.

LEMMA 3.1.10. *Let $p : \mathbf{F} \to \mathbf{C}$ be a fibration in groupoids. Let*

$$\begin{tikzcd}
a_0 \arrow[r] \arrow[d] & a_1 & & S_0 \arrow[r] \arrow[d] & S_1 \\
a_2 & & \textit{over} & S_2 &
\end{tikzcd}$$

be a diagram in $\mathbf{F}$, resp. $\mathbf{C}$. Assume that the fibred sums $a := a_1 \amalg_{a_0} a_2$ and $S := S_1 \amalg_{S_0} S_2$ exist. Then the following hold.

(1) *$p(a) = S$*
(2) *If the diagram*

$$\begin{tikzcd}
a_0 \arrow[r] \arrow[d] & a_1 \arrow[d, "\tilde{f}_1"'] & & S_0 \arrow[r] \arrow[d] & S_1 \arrow[d, "f_1"'] \\
a_2 \arrow[r, "\tilde{f}_2"] & b & \textit{maps to} & S_2 \arrow[r, "f_2"] & S
\end{tikzcd}$$

then there is an isomorphism $b \cong a$ over id_S.

PROOF. Let $\tilde{g}_i : a_i \to a$ be the canonical morphism into the sum and $g_i := p(\tilde{g}_i) : S_i \to p(a)$. By the universal property of fibred sums there is a unique morphism $g = g_1 \amalg g_2 : S \to p(a)$ inducing g_i on S_i. Consider $a' := g^*(a) = S \times_{p(a)} a$. Since the morphisms in $\mathbf{F}$ are cartesian the diagram of solid arrows

$$\begin{tikzcd}
a_0 \arrow[rr] \arrow[dd] & & a_1 \arrow[dl, dotted] \arrow[dd, "\tilde{g}_1"] & & S_0 \arrow[rr] \arrow[dd] & & S_1 \arrow[dl] \arrow[dd, "g_1"] \\
& a' \arrow[dr, "v"] & & \text{over} & & S \arrow[dr] & \\
a_2 \arrow[ur, dotted] \arrow[rr, "\tilde{g}_2"'] & & a & & S_2 \arrow[ur] \arrow[rr, "g_2"'] & & p(a)
\end{tikzcd}$$

can be completed as indicated by the dotted arrows, where v is the canonical morphism. By the universal property of $a = a_1 \amalg_{a_0} a_2$ there is a morphism $w : a \to a'$ with $v \circ w = id_a$. On the other hand, by the universal property of $S = p(a')$ one has $p(w) \circ p(v) = id_S$. Hence $p(u), p(w)$ are isomorphisms and so are u, v as p detects isomorphisms. This proves (1).

In order to show (2) consider the morphism $\tilde{f} := \tilde{f}_1 \amalg \tilde{f}_2 : a \to b$. Applying p and using (1) we obtain $p(\tilde{f}) = f_1 \amalg f_2 : S \to S$. As $f_1 \amalg f_2$ is the identity on S (2) follows. □

We will now introduce deformation theories which are our central objects of study. To investigate deformations over extensions of complex spaces it is important to know about the existence of certain fibred sums. More precisely the following condition is important – and satisfied in all our examples 3.1.6, see Section ?????.

Definition 3.1.11 (Homogeneity). Let $p : \mathbf{F} \to \mathbf{An}_\Sigma$ be a fibration in groupoids. Then p is called a *homogeneous fibration*, or a *deformation theory* in brief, if the following condition is satisfied:

(H): Let

$$\begin{tikzcd} a \arrow[r] \arrow[d] & a' \\ b & \end{tikzcd} \quad \text{be a diagram over} \quad \begin{tikzcd} S \arrow[r] \arrow[d] & S' \\ T & \end{tikzcd}$$

such that $S \to T$ is finite and $S \hookrightarrow S'$ is an extension by a coherent $\mathcal{O}_S$-module $\mathcal{M}$. Then there exists the fibred sum $b' := a' \amalg_a b$.

Example 3.1.12. Schuster's result 2.4.4 implies that the examples of 3.1.6 are deformation theories.

We introduce a similar notation for functors.

Definition 3.1.13. (1) Let $G : \mathbf{An}_\Sigma \to (\mathbf{Sets})$ be a functor and let $g : \mathbf{F} \to \mathbf{An}_\Sigma$ be the associated fibration in groupoids, see 3.1.2. Then G is called a *homogeneous functor* if g is homogeneous.

(2) A functor of isomorphism classes $[\mathbf{F}]$ associated to a deformation theory $p : \mathbf{F} \to \mathbf{An}_\Sigma$ is called the *deformation functor* underlying p.

Remarks 3.1.14. (1) Let $p : \mathbf{F} \to \mathbf{An}_\Sigma$ be a deformation theory and $[\mathbf{F}] : \mathbf{An}_\Sigma \to \mathbf{Sets}$ be the associated functor of isomorphism classes. In general $[\mathbf{F}]$ is *not* homogeneous as the example 3.1.15 below shows.

(2) In the literature also the somewhat weaker notion of semihomogeneity is studied, see [**Schu**], [**Rim**]. A fibration in groupoids is called *semihomogeneous* if the following two conditions — introduced by Schlessinger — are satisfied.

(S1a): In the situation of (H) in 3.1.11 the fibred sum $a' \amalg_a b$ exists if $S_{red} \hookrightarrow T$ is a closed embedding and $S \hookrightarrow S'$ is a trivial extension by a coherent $\mathcal{O}_S$-module.

(S1b): In the situation of (H), if $S_{red} \hookrightarrow T$ is a closed embedding and $S \to S'$ is any extensions then there is a commutative diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d] & a' \arrow[d] \\ b \arrow[r] & b' \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} S \arrow[r] \arrow[d] & S' \arrow[d] \\ T \arrow[r] & T' := S' \amalg_S T \end{tikzcd}$$

(where b' is not necessarily the fibred sum).

As above, a functor $G : \mathbf{F} \to (\mathbf{Sets})$ will be called *semihomogeneous* if the associated groupoid is semihomogeneous. The reader may easily verify that for a semihomogeneous fibration in groupoids $p : \mathbf{F} \to \mathbf{An}_\Sigma$ the associated functor of isomorphism classes $[\mathbf{F}]$ is again semihomogeneous. Since in all the applications the semihomogeneous functors arise as deformation functors, i.e. as functors of isomorphism classes of homogeneous fibrations in groupoids, we will restrict ourselves to the study of deformation theories as introduced above. In general, a deformation functor is not homogeneous as is seen by the subsequent example. However, we will see later that for deformation theories with a certain automorphism lifting property the associated functor is again homogeneous. The role of infinitesimal automorphisms can be already seen in the following example.

Example 3.1.15. Consider the cocartesian diagram of complex spaces

$$\begin{tikzcd}
T' := T[\mathbb{C}\varepsilon] & S_2 \arrow[l, hook'] \\
T := (\mathbb{C}, 0) \arrow[u] & S_1 \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

where S_i is the fat point with $\mathcal{O}_{S_i} = \mathbb{C}\{s\}/(s^{i+1})$. Here the map $T \hookrightarrow T[\mathbb{C}\varepsilon]$ is the canonical inclusion, and $S_2 \to T[\mathbb{C}\varepsilon]$ is given on the level of structure sheaves $\mathbb{C}\{t\}[\varepsilon] \to \mathbb{C}\{s\}/(s^3)$ by $t \mapsto s, \quad \varepsilon \mapsto s^2$. Let $X, Y \subseteq T' \times \mathbb{C}$ be given by the equations

$$X := \{u^2 - t = 0\} \quad \text{and} \quad Y := \{u^2 - \frac{2\varepsilon}{(1+t)^2} - t = 0\},$$

where u denotes the coordinate function of the second factor of $T' \times \mathbb{C}$. Then X, Y are flat 2-fold coverings of T'. Obviously X, Y are not T'-isomorphic since they are even not isomorphic modulo t, i.e. when restricting to the double point $\{t = 0\}$ in T'. On the other hand, the families

$$X \times_{T'} T, \quad Y \times_{T'} T$$

are T-isomorphic, since T is given by $\{\varepsilon = 0\}$. The restrictions to S_2 are given by

$$X \times_{T'} S_2 = \{u^2 - s = 0\} \quad \text{and} \quad Y \times_{T'} S_2 = \{u^2 + \frac{2s^2}{(1+s)^2} - s = 0\}.$$

These spaces are S_2-isomorphic. In fact, after the coordinate transformation $u = u'(1+s)$ the space $X \times_{T'} S_2$ is given by the equation

$$0 = u'^2(1+s)^2 - s = u'^2 + 2su'^2 + s^2u'^2 - s = u'^2 + \frac{2s^2}{(1+s)^2} - s,$$

since $u'^2 = s/(1+s)^2$ and hence $s^2u'^2 = 0 \bmod s^3$. Thus the map $(u, s) \mapsto (u(1+s), s)$ yields an isomorphism from $Y \times_{T'} S_2$ onto $X \times_{T'} S_2$.

3.2. The exact sequences of cofibrations in groupoids

Let S be a fixed complex space over $\Sigma \in$ **An**. The aim of this section is to derive the two basic exact sequences of deformation theory. For instance the first main result implies the following proposition.

Proposition 3.2.1. *For every exact sequence of* $\mathcal{O}_S$*-modules* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *there is long exact sequence of* $\Gamma(S, \mathcal{O}_S)$*-modules*

$$\begin{aligned} 0 \longrightarrow \operatorname{Der}_\Sigma(\mathcal{O}_S, \mathcal{M}') \longrightarrow \operatorname{Der}_\Sigma(\mathcal{O}_S, \mathcal{M}) \longrightarrow \operatorname{Der}_\Sigma(\mathcal{O}_S, \mathcal{M}'') \\ \xrightarrow{\ \delta\ } \operatorname{Ex}_\Sigma(S, \mathcal{M}') \longrightarrow \operatorname{Ex}_\Sigma(S, \mathcal{M}) \longrightarrow \operatorname{Ex}_\Sigma(S, \mathcal{M}'') \ . \end{aligned}$$

Moreover, δ *is* ∂*-functorial.*

We will derive this as a special case of theorem 3.2.2 below where we consider the following situation.

Let $p : \mathbf{G} \longrightarrow \mathbf{Coh}(S)$ be a cofibration in groupoids and assume that $\mathbf{G}(0) = \{e\}$ consists of one object with $\operatorname{Mor}(e) = \{id_e\}$. For a coherent $\mathcal{O}_S$-module $\mathcal{M}$ we will denote by $e[\mathcal{M}]$ the object $i_*(e)$ where $i : 0 \to \mathcal{M}$ is the zero map. Obviously for any morphism of coherent $\mathcal{O}_S$-modules $\varphi : \mathcal{M} \to \mathcal{N}$ we have $\varphi_* \left(e[\mathcal{M}] \right) = e[\mathcal{N}]$. By

$$\mathrm{Aut}\left(e[\mathcal{M}] \right)$$

we denote the set of isomorphisms of $e[\mathcal{M}]$ in $\mathbf{G}(\mathcal{M})$, i.e. those lying over $id_{\mathcal{M}}$. By the universal property of i_* obviously

$$\mathrm{Aut}\left(e[\mathcal{M}] \right) \cong \mathrm{Hom}_{\mathbf{G}}\left(e, e[\mathcal{M}] \right) \ .$$

The set of isomorphism classes of $\mathbf{G}(\mathcal{M})$ will be denoted by $G(\mathcal{M})$, i.e. $G(\mathcal{M}) = [\mathbf{G}(\mathcal{M})]$ in our previous notation. The first main result of this section is the following theorem.

Theorem 3.2.2. *Assume that there exist fibred products in* $\mathbf{G}$. *Then the following hold.*

(1) *For every coherent* $\mathcal{O}_S$*-module* $\mathcal{M}$ *the sets* $G(\mathcal{M})$ *and* $\mathrm{Aut}\left(e[\mathcal{M}] \right)$ *carry natural* $\mathcal{O}_S$*-module structures such that* $\mathcal{M} \to G(\mathcal{M})$ *and* $\mathcal{M} \mapsto \mathrm{Aut}\left(e[\mathcal{M}] \right)$ *are functors* $\mathbf{Coh}(S) \to \mathbf{Mod}\left(\Gamma(S, \mathcal{O}_S) \right)$.

(2) *Let* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *be an exact sequence of coherent* $\mathcal{O}_S$*-modules. Then there is an induced natural exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{Aut}\left(e[\mathcal{M}'] \right) \longrightarrow \mathrm{Aut}\left(e[\mathcal{M}] \right) \longrightarrow \mathrm{Aut}\left(e[\mathcal{M}''] \right) \\ \longrightarrow G(\mathcal{M}') \longrightarrow G(\mathcal{M}) \longrightarrow G(\mathcal{M}'') \ . \end{aligned}$$

Before embarking on the proof of 3.2.2 we show how 3.2.1 follows.

3.2.3. Let $\mathbf{Ex}_\Sigma(S)$ be the category of extensions of S over Σ (see 2.4.1) and

$$\mathbf{G} := \mathbf{Ex}_\Sigma(S)^0 \longrightarrow \mathbf{Coh}(S),$$

the natural functor associating to an extension $(S \hookrightarrow T, u)$ of S by the coherent $\mathcal{O}_S$-module $\mathcal{M}$ the underlying module $\mathcal{M} \in \mathbf{Coh}(S)$. That this is a cofibration in groupoids follows from the discussion in 2.4.1. Moreover, by 2.4.11 there are fibred sums in $\mathbf{Ex}_\Sigma(S)$ or, equivalently, fibred products in $\mathbf{Ex}_\Sigma(S)^0$. Using 2.3.9 we get that $\mathrm{Aut}(S[\mathcal{M}])$ is canonically isomorphic to $\mathrm{Der}_\Sigma(\mathcal{O}_S, \mathcal{M})$. By the definitions, $G(\mathcal{M})$ is just $\mathrm{Ex}_\Sigma(S, \mathcal{M})$. Applying 3.2.2 the proposition follows.

In order to show 3.2.2 we will proceed in a series of lemmata. For the existence of natural $\Gamma(S, \mathcal{O}_S)$-module structures on $\mathrm{Aut}(e[\mathcal{M}])$ and $G(\mathcal{M})$ we will apply 2.4.4. Therefore we must show that these functors are compatible with products. Let $p : \mathbf{G} \to \mathbf{Coh}(S)$ be as in 3.2.2.

Lemma 3.2.4. *Let* $\varphi_1 : \mathcal{M}_1 \to \mathcal{M}_0$ *and* $\varphi_2 : \mathcal{M}_2 \to \mathcal{M}_0$ *be homomorphisms of coherent* $\mathcal{O}_S$*-modules and set* $\mathcal{M} := \mathcal{M}_1 \times_{\mathcal{M}_0} \mathcal{M}_2$. *For an element* $g \in \mathbf{G}(\mathcal{M})$ *let* g_i *be the induced element in* $\mathbf{G}(\mathcal{M}_i)$. *Then the natural map*

$$\mathrm{Aut}(g) \longrightarrow \mathrm{Aut}(g_1) \times_{\mathrm{Aut}(g_0)} \mathrm{Aut}(g_2)$$

is bijective.

Proof. The diagram

$$\begin{tikzcd} g \arrow[r] \arrow[d] & g_1 \arrow[d] & & \mathcal{M} \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & \mathcal{M}_1 \arrow[d, "\varphi_1"] \\ g_2 \arrow[r] & g_0 & \text{over} & \mathcal{M}_2 \arrow[r, "\varphi_2"] & \mathcal{M}_0 \end{tikzcd}$$

in **G** is cartesian by the dual version of 3.1.10 (2). Thus, if $\alpha_i : g_i \to g_i$ are morphisms over $id_{\mathcal{M}_i}$ with $\alpha_0 = \varphi_{i*}(\alpha_i)$, $i = 1, 2$, then by the universal property of the fibre product there is a *unique* morphism $\alpha : g \to g$ with $\psi_{i*}(\alpha) = \alpha_i$. This proves the lemma. ☐

For the functor G of isomorphism classes of **G** we get a weaker statement.

Lemma 3.2.5. *The natural map*

$$\varphi_* : G(\mathcal{M}) \longrightarrow G(\mathcal{M}_1) \times_{G(\mathcal{M}_0)} G(\mathcal{M}_2)$$

given by $\varphi_*(g) = (\varphi_{1*}(g), \varphi_{2*}(g))$ is surjective. Moreover, if $\mathcal{M}_0 = 0$ then this map is even bijective.

Proof. That φ_* is surjective follows immediately from the existence of fibre products in **G**. Now assume that $\mathcal{M}_0 = 0$. Then $G(0) = \{e\}$, and for $g_i \in G(\mathcal{M}_i)$ there is a unique morphism $g_i \to e$. On the other hand, if $g \in G(\mathcal{M})$ with $\varphi_*(g) = (g_1, g_2)$ then g is given as a fibre product

$$\begin{tikzcd} g := g_1 \times_{(\alpha_1,\alpha_2)} g_2 \arrow[r] \arrow[d] & g_2 \arrow[d, "\alpha_2"] \\ g_1 \arrow[r, "\alpha_1"] & e \end{tikzcd}$$

for some pair of morphisms $(\alpha_1, \alpha_2) \in \mathrm{Hom}(g_1, e) \times \mathrm{Hom}(g_2, e)$. This proves the lemma. ☐

Corollary 3.2.6. *The sets* $\mathrm{Aut}\,(e[\mathcal{M}])$ *and* $G(\mathcal{M})$ *carry natural* $\Gamma(S, \mathcal{O}_S)$-*module structures. Moreover, the functor*

$$\mathcal{M} \longrightarrow \mathrm{Aut}\,(e[\mathcal{M}])$$

is left exact and

$$\mathcal{M} \longrightarrow G(\mathcal{M})$$

is half exact.

Proof. That the sets $\mathrm{Aut}\,(e[\mathcal{M}])$ and $G(\mathcal{M})$ carry natural $\Gamma(S, \mathcal{O}_S)$-module structures, follows from 2.4.4, 3.2.4 and 3.2.5. Now assume that

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

is an exact sequence of $\mathcal{O}_S$-modules. Then $\mathcal{M}' = \mathcal{M} \times_{\mathcal{M}''} 0$ which implies that

$$\mathrm{Aut}\,(e[\mathcal{M}']) \cong \mathrm{Aut}\,(e[\mathcal{M}]) \times_{\mathrm{Aut}(e[\mathcal{M}''])} \mathrm{Aut}(e).$$

As $\mathrm{Aut}(e) = 0$ this shows that $\mathrm{Aut}\,(e[\mathcal{M}'])$ is the kernel of the map $\mathrm{Aut}\,(e[\mathcal{M}]) \to \mathrm{Aut}\,(e[\mathcal{M}''])$. Similarly, the surjectivity of

$$G(\mathcal{M}') \longrightarrow G(\mathcal{M}) \times_{G(\mathcal{M}'')} G(0)$$

gives that $G(\mathcal{M}') \to G(\mathcal{M}) \to G(\mathcal{M}'')$ is exact. ☐

Our next task is to define the connecting homomorphism δ in 3.2.2 (2). Let

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \xrightarrow{q} \mathcal{M}'' \longrightarrow 0$$

be an exact sequence of $\mathcal{O}_S$-modules. We will define the functorial map

$$\mathrm{Aut}\,(e[\mathcal{M}'']) \xrightarrow{\delta} G(\mathcal{M}')$$

by the following construction. First we observe that the canonical map

$$A := \operatorname{Aut}(e[\mathcal{M}'']) \longrightarrow \operatorname{Mor}(e[\mathcal{M}], e[\mathcal{M}''])$$

given by composing with the natural map $e[q] : e[\mathcal{M}] \to e[\mathcal{M}'']$ is bijective, since $\mathbf{G} \to \mathbf{Mod}(S)$ is cofibred in groupoids. For a morphism $\alpha \in A$ we set

$$\delta(\alpha) := \left[e[\mathcal{M}] \times_{\alpha \circ e[q]} e \right] .$$

We will show:

Lemma 3.2.7. (1) *δ is a homomorphism of $\Gamma(S, \mathcal{O}_S)$-modules.*
(2) *δ is ∂-functorial, i.e. functorial in morphisms of exact sequences.*
(3) *The sequence*

$$\operatorname{Aut}(e[M]) \longrightarrow \operatorname{Aut}(e[M'']) \xrightarrow{\ \delta\ } G(M') \longrightarrow G(M)$$

is exact.

Proof. We remind the reader, that the ∂-functoriality means that for every commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}' \arrow[r] \arrow[d] & \mathcal{M} \arrow[r, "q"] \arrow[d] & \mathcal{M}'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & \mathcal{N}' \arrow[r] & \mathcal{N} \arrow[r, "q"] & \mathcal{N}'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows the induced diagram

$$\begin{tikzcd}
\operatorname{Aut}(e[\mathcal{M}'']) \arrow[r, "\delta"] \arrow[d] & G(\mathcal{M}') \arrow[d] \\
\operatorname{Aut}(e[\mathcal{N}'']) \arrow[r, "\delta"] & G(\mathcal{N}')
\end{tikzcd}$$

is commutative. But this follows easily from the diagram

$$\begin{tikzcd}
& e \arrow[rr] \arrow[dd, equal] & & e[\mathcal{M}''] \arrow[dd] \\
e[\mathcal{M}] \times_{\alpha \circ e[q]} e \arrow[ur] \arrow[rr] \arrow[dd, dotted] & & e[\mathcal{M}] \arrow[ur, "\alpha \circ e[q]"] \arrow[dd] & \\
& e \arrow[rr] & & e[\mathcal{N}''] \\
e[\mathcal{N}] \times_{\beta \circ e[q]} e \arrow[ur] \arrow[rr] & & e[\mathcal{N}], \arrow[ur, "\beta \circ e[q]"'] &
\end{tikzcd}$$

where $\beta := \varphi''_*(\alpha)$.

(1) is a consequence of (2) and 2.4.4 (2).

Next we will show that

$$\operatorname{Aut}(e[\mathcal{M}'']) \xrightarrow{\ \delta\ } G(\mathcal{M}') \xrightarrow{\ i_*\ } G(\mathcal{M})$$

is exact. Obviously the composition of these two maps is zero. Conversely, consider $[g'] \in G(\mathcal{M}')$ with $i_*(g') \cong e[\mathcal{M}]$. This means that there is a morphism $\alpha : g' \to$

$e[\mathcal{M}]$ over i. Composing with $e[\mathcal{M}] \to e[\mathcal{M}'']$ gives a morphism $g' \to e[\mathcal{M}'']$ lying over the zero map. Thus it factors through e, and we get a commutative diagram

$$\begin{tikzcd} g' \arrow[r] \arrow[d] & e[\mathcal{M}] \arrow[d] \\ e \arrow[r] & e[\mathcal{M}'']. \end{tikzcd}$$

By the dual version of 3.1.10 the diagram is cartesian, whence $g' = \delta(\alpha)$.

Finally we prove that

$$\operatorname{Aut}(e[\mathcal{M}]) \longrightarrow \operatorname{Aut}(e[\mathcal{M}'']) \longrightarrow G(\mathcal{M}')$$

is exact. Assume that $\alpha = q_*(\beta)$ for some $\beta \in \operatorname{Aut}(e[\mathcal{M}])$, i.e. $\alpha \circ e[q] = e[q] \circ \beta$. Then $\delta(\alpha)$ is represented by $e[\mathcal{M}] \times_{e[q]\circ\beta} e$. The morphism

$$e[\mathcal{M}] \times_{e[q]\circ\beta} e \cong (e[\mathcal{M}] \times_{e[q]} e) \times_{\beta} e[\mathcal{M}] \xrightarrow{proj} e[\mathcal{M}] \times_{e[q]} e.$$

is lying over $id_{\mathcal{M}'}$ and so is an isomorphism. As $e[\mathcal{M}] \times_{e[q]} e \cong e[\mathcal{M}']$ we get that $\delta(\alpha) = 0$.

Conversely, if $\alpha \in \operatorname{Aut}(e[\mathcal{M}''])$ is given with $\delta(\alpha) = 0$ then there is an isomorphism $e[\mathcal{M}] \times_{\alpha\circ e[q]} e \cong e[\mathcal{M}']$. Taking i_* gives an isomorphism $\beta : e[\mathcal{M}] \to e[\mathcal{M}]$. It is easily seen that $q_*(\beta) = \alpha$. ☐

In the rest of this section we will derive the second important sequence of deformation theory. It will be used in the next section to derive the so called Kodaira Spencer sequence associated to deformation theories. We consider the following setup. Let

$$\begin{tikzcd} \mathbf{F} \arrow[rr, "\sigma"] \arrow[dr, "p"'] & & \mathbf{G} \arrow[dl, "q"] \\ & \mathbf{Mod}(S) & \end{tikzcd}$$

be a morphism of cofibred groupoids over $\mathbf{Mod}(S)$. We always assume that the fibers

$$\mathbf{F}(0) = \{e_F\}, \qquad \mathbf{G}(0) = \{e_G\}$$

are just the trivial categories. For $\mathcal{M} \in \mathbf{Coh}(S)$ we denote as above by

$$e_F[\mathcal{M}] \qquad \text{resp.} \qquad e_G[\mathcal{M}]$$

the objects $i_*(e_F)$, $i_*(e_G)$, where $i : 0 \hookrightarrow \mathcal{M}$ is the natural map. These constructions are functorial in $\mathcal{M}$. We define the kernel of σ denoted by $\mathbf{K} = \mathbf{Kern}(\sigma)$ to be the following subcategory of $\mathbf{F}$: The objects of $\mathbf{K}$ are those objects $a \in \mathbf{F}$ with $\sigma(a) = e_G[\mathcal{M}]$ whee $\mathcal{M} := p(a)$. The morphism $a \xrightarrow{\alpha} b$ in $\mathbf{K}$ over $\mathcal{M} \xrightarrow{p(\alpha)} \mathcal{N}$ are those morphisms in $\mathbf{F}$ for which

$$e_G[\mathcal{M}] \xrightarrow{\sigma(\alpha)} e_G[\mathcal{N}]$$

is the canonical map induced by $p(\alpha)$, i.e. $\sigma(\alpha) = e_G[p(\alpha)]$. Observe that $\mathbf{K}$ is *not* in general a *full* subcategory of $\mathbf{F}$!

Since $\sigma(e_F[\mathcal{M}]) = e_G[\mathcal{M}]$ we have always $e_F[\mathcal{M}] \in \mathbf{K}$. In order to clarify the notation we will write $e_K[\mathcal{M}]$ if we consider this as an element of $\mathbf{K}$.

In this situation, the Kodaira Spencer map

$$\delta : \operatorname{Aut}(e_G[\mathcal{M}]) \longrightarrow K(\mathcal{M})$$

is given by associating to $\alpha \in \mathrm{Aut}\,(e_G[\mathcal{M}])$, the class

$$\delta(\alpha) := [\alpha_*\,(e_F[\mathcal{M}])] \in K(\mathcal{M})$$

that fits into the following diagram

$$\begin{tikzcd}
e_F \arrow[r] \arrow[d, maps to] & e_F[\mathcal{M}] \arrow[r] \arrow[d, maps to] & \alpha_*\,(e_F[\mathcal{M}]) \arrow[d, maps to] \\
e_G \arrow[r] & e_G[\mathcal{M}] \arrow[r, "\alpha"] & e_G[\mathcal{M}]
\end{tikzcd}$$

whose squares are cocartesian.

With these notations we show the following result.

Theorem 3.2.8. *For every coherent $\mathcal{O}_S$-module $\mathcal{M}$ there is a natural exact sequence*

$$0 \to \mathrm{Aut}\,(e_K[\mathcal{M}]) \to \mathrm{Aut}\,(e_F[\mathcal{M}]) \to \mathrm{Aut}(e_G[\mathcal{M}]) \xrightarrow{\delta}$$
$$K(\mathcal{M}) \to F(\mathcal{M}) \to G(\mathcal{M}).$$

Proof. It is a direct consequence of the definitions that the composition of any two consecutive maps in the above sequence is the zero map. That the sequence

$$0 \to \mathrm{Aut}\,(e_K[\mathcal{M}]) \to \mathrm{Aut}\,(e_F[\mathcal{M}]) \to \mathrm{Aut}\,(e_G[\mathcal{M}])$$

is exact, follows from the definition of $\mathrm{Aut}\,(e_K[\mathcal{M}])$. The sequence

$$K(\mathcal{M}) \longrightarrow F(\mathcal{M}) \xrightarrow{\sigma_*} G(\mathcal{M})$$

is exact: for $[a] \in F(\mathcal{M})$, which is in the kernel of σ_* we have an isomorphism $\varphi : \sigma(a) \cong e_G[\mathcal{M}]$. Since σ is a cofibration there is an isomorphism $a \to a'$ over φ, and then $[a'] \in K(\mathcal{M})$ maps to $[a]$ in $F(\mathcal{M})$.

If $\delta(\alpha) = 0$ then $\alpha_* e_F[\mathcal{M}] \cong e_F[\mathcal{M}]$ in $\mathbf{K}(\mathcal{M})$, and the composition with the canonical map $e_F[\mathcal{M}] \to \alpha_* e_F[\mathcal{M}]$ yields an automorphism of $e_F[\mathcal{M}]$ in $\mathbf{F}$ over α. Finally, if $[b] \in K(\mathcal{M})$ maps to zero in $F(\mathcal{M})$ then there is an isomorphism $\beta : e_F[\mathcal{M}] \to b$ in $\mathbf{F}$. Applying σ we get a cocartesian diagram

$$\begin{tikzcd}
e_F[\mathcal{M}] \arrow[r, "\beta"] \arrow[d, maps to] & b \arrow[d, maps to] \\
e_G[\mathcal{M}] \arrow[r, "\alpha:=\sigma(\beta)"] & e_G[\mathcal{M}] = \sigma(b)
\end{tikzcd}$$

showing that $b \cong \alpha_*\,(e_F[\mathcal{M}])$. □

Remark 3.2.9. Later on we will see that there are naturally defined cohomology functors $T^i_{S/\Sigma}(\mathcal{M})$, $i \geq 0$, such that

(1) $T^0_{S/\Sigma}(\mathcal{M}) \cong \mathrm{Der}_\Sigma(\mathcal{O}_S, \mathcal{M})$,

(2) $T^1_{S/\Sigma}(\mathcal{M}) \cong \mathrm{Ex}_\Sigma(S, \mathcal{M})$,

(3) every exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ induces a long exact cohomology sequence

$$0 \to T^0_{S/\Sigma}(\mathcal{M}') \to T^0_{S/\Sigma}(\mathcal{M}) \to T^0_{S/\Sigma}(\mathcal{M}'') \to T^1_{S/\Sigma}(\mathcal{M}') \to \cdots$$

extending the exact sequence in 3.2.1.

We do not know whether one can also extend the exact sequence of 3.2.2 to an exact cohomology sequence in a similar way, i.e. whether in the abstract setting of 3.2.2 one can define suitable right derived functors of $\mathcal{M} \mapsto \mathrm{Aut}(e[\mathcal{M}])$ such that the first derived functor coincides with $G(\mathcal{M})$. Similarly one can ask whether one can extend the sequence 3.2.8 to the right.

3.3. Infinitesimal Extensions and the Kodaira-Spencer Sequence

We will now apply the results of the previous section to introduce two type of extension modules associated to deformation theories. In particular we will study the Kodaira Spencer map which is a special case of the sequence in 3.2.8 and which is of fundamental importance. At the end of the section we will make the Kodaira Spencer map more explicit for deformations of complex spaces and give a homological description.

We start by introducing the category of extensions $\mathbf{Ex}_\Sigma(a)$ of a given element a of a deformation theory.

3.3.1. Let Σ be a fixed complex space and $p : \mathbf{F} \to \mathbf{An}_\Sigma$ be a deformation theory. For $S \in \mathbf{An}_\Sigma$, $a \in \mathbf{F}(S)$ and a coherent $\mathcal{O}_S$-module $\mathcal{M}$ we consider extensions of a by $\mathcal{M}$, i.e. pairs $(a \to b, u)$ such that the underlying morphism $S \to T := p(b)$ is an extension of S by$\mathcal{M}$, with a fixed isomorphism of $\mathcal{O}_S$-modules

$$\mathcal{M} \xrightarrow{u} \operatorname{Ker}(\mathcal{O}_T \to \mathcal{O}_S).$$

Now assume that $\mathcal{M}'$ is another coherent $\mathcal{O}_S$-module and $(a \hookrightarrow b', u')$ is an extension of a by $\mathcal{M}'$. Then a morphism

$$(a \hookrightarrow b, u) \to (a \hookrightarrow b', u')$$

consists in a morphism $\beta : b \to b'$ making the diagram

$$\begin{tikzcd}
a \arrow[r] \arrow[d, equal] & b \arrow[d, "\beta"] \\
a \arrow[r] & b'
\end{tikzcd}$$

commutative. Obviously $\beta : b \to b'$ induces a morphism of extensions $p(\beta) : T \to T'$.

The extensions of a by coherent $\mathcal{O}_S$-modules $\mathcal{M}$ form a category $\mathbf{Ex}_\Sigma(a)$, which fibers over $\mathbf{Ex}_\Sigma(S)$, the category of extensions of S by coherent $\mathcal{O}_S$-modules, see **??**. This is indeed a fibration, since over a morphism

$$(T', u') \xrightarrow{\gamma} (T, u)$$

of extensions of S by $\mathcal{M}'$ resp. $\mathcal{M}$ and an object $(a \hookrightarrow b, u)$ there is always a cartesian morphism over γ, namely $(a \hookrightarrow \gamma^*(b), u')$. Clearly a morphism in $\mathbf{Ex}_\Sigma(a)$ is an isomorphism iff the underlying morphism in $\mathbf{Ex}_\Sigma(S)$ is an isomorphism. Hence

$$\mathbf{Ex}_\Sigma(a) \to \mathbf{Ex}_\Sigma(S)$$

is a fibration in groupoids. As we saw earlier, also

$$\mathbf{Ex}_\Sigma(S) \to \mathbf{Coh}(S)^0$$

is a fibration in groupoids. Hence we get a commutative diagram of cofibrations

$$\begin{tikzcd}
\mathbf{F} := \mathbf{Ex}_\Sigma(a)^0 \arrow[rr] \arrow[dr] & & \mathbf{G} := \mathbf{Ex}_\Sigma(S)^0 \arrow[dl] \\
& \mathbf{Coh}(S). &
\end{tikzcd}$$

We let $\mathbf{K} := \mathbf{Ker}(\mathbf{F} \to \mathbf{G})$ be the kernel of the above functor, see 3.2.8. There are the distinguished elements $e_{\mathbf{F}}[\mathcal{M}]$ in $\mathbf{F}$ resp. $e_{\mathbf{K}}[\mathcal{M}]$ in $\mathbf{K}$ which are obviously represented by the trivial extension $a[\mathcal{M}]$ of a. Here by $a[\mathcal{M}]$ we denote in brief the extension $(a \hookrightarrow a[\mathcal{M}], i_{\mathcal{M}})$ where $i_{\mathcal{M}}$ is the canonical injection $\mathcal{M} \hookrightarrow \mathcal{O}_S[\mathcal{M}]$. For a coherent $\mathcal{O}_S$-module $\mathcal{M}$ we set (using the notations of the previous sections)

$$\begin{aligned} \operatorname{Aut}_\Sigma(a/S, \mathcal{M}) &:= \operatorname{Aut}_{\mathbf{K}}(a[\mathcal{M}]), & \operatorname{Aut}_\Sigma(a, \mathcal{M}) &:= \operatorname{Aut}_{\mathbf{F}}(a[\mathcal{M}]), \\ \operatorname{Ex}_\Sigma(a/S, \mathcal{M}) &:= K(\mathcal{M}), & \operatorname{Ex}_\Sigma(a, \mathcal{M}) &:= F(\mathcal{M}). \end{aligned}$$

3.3.2. More explicitly, these objects can be described as follows. An element of $\operatorname{Ex}_\Sigma(a, \mathcal{M})$ is given by the isomorphism class of a pair $(a \hookrightarrow b, u)$, and two elements $(a \hookrightarrow b, u)$ and $(a \hookrightarrow b', u')$ are isomorphic iff there exists an isomorphism $\beta : b \to b'$ compatible with the morphism $a \hookrightarrow b, a \hookrightarrow b'$ and such that

$$p(\beta) : T := p(b) \longrightarrow T' := p(b')$$

fits into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M} \arrow[r, "u'"] \arrow[d, equal] & \mathcal{O}_{T'} \arrow[r] \arrow[d, "p(\beta)^*"] & \mathcal{O}_S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r, "u"] & \mathcal{O}_T \arrow[r] & \mathcal{O}_S \arrow[r] & 0.
\end{tikzcd}$$

Moreover $\operatorname{Aut}_\Sigma(a, \mathcal{M})$ is the set of all such isomorphisms of the pair $(a \hookrightarrow a[\mathcal{M}], i_{\mathcal{M}})$ into itself.

Similarly, the elements of $\operatorname{Ex}_\Sigma(a/S, \mathcal{M})$ are represented by pairs $(a \hookrightarrow b, i_{\mathcal{M}})$ such that $p(b) = S[\mathcal{M}]$. Moreover, two pairs $(a \hookrightarrow b, i_{\mathcal{M}})$ and $(a \hookrightarrow b', i_{\mathcal{M}})$ give the same element in $\operatorname{Ex}_\Sigma(a/S, \mathcal{M})$ iff there is an isomorphism $b \to b'$ inducing the identity on $p(b) = S[\mathcal{M}]$ and compatible with the maps $a \hookrightarrow b$ and $a \hookrightarrow b'$. The elements of $\operatorname{Aut}_\Sigma(a/S, \mathcal{M})$ are just all such isomorphisms of the pair $(a \hookrightarrow a[\mathcal{M}], i_{\mathcal{M}})$ into itself.

Applying 3.2.2 to the above diagram of cofibrations we get the following result.

THEOREM 3.3.3. (1) $\mathbf{Ex}_\Sigma(a)$ *admits fibred direct products.*

(2) *The sets* $\operatorname{Aut}_\Sigma(a, \mathcal{M})$, $\operatorname{Aut}_\Sigma(a/S, \mathcal{M})$, $\operatorname{Ex}_\Sigma(a, \mathcal{M})$ *and* $\operatorname{Ex}_\Sigma(a/S, \mathcal{M})$ *carry natural* $\Gamma(S, \mathcal{O}_S)$*-module structures. Moreover, the Aut-modules are compatible with fibred products and the Ex-modules with finite direct products.*

(3) *For every exact sequence of coherent* $\mathcal{O}_S$*-modules* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *there are exact sequences of* $\Gamma(S, \mathcal{O}_S)$*-modules*

$$\begin{aligned} 0 \longrightarrow \operatorname{Aut}_\Sigma(a, \mathcal{M}') \longrightarrow \operatorname{Aut}_\Sigma(a, \mathcal{M}) \longrightarrow \operatorname{Aut}_\Sigma(a, \mathcal{M}'') \longrightarrow \\ \operatorname{Ex}_\Sigma(a, \mathcal{M}') \longrightarrow \operatorname{Ex}_\Sigma(a, \mathcal{M}) \longrightarrow \operatorname{Ex}_\Sigma(a, \mathcal{M}''). \end{aligned}$$

and

$$\begin{aligned} 0 \longrightarrow \operatorname{Aut}_\Sigma(a/S, \mathcal{M}') \longrightarrow \operatorname{Aut}_\Sigma(a/S, \mathcal{M}) \longrightarrow \operatorname{Aut}_\Sigma(a/S, \mathcal{M}'') \longrightarrow \\ \operatorname{Ex}_\Sigma(a/S, \mathcal{M}') \longrightarrow \operatorname{Ex}_\Sigma(a/S, \mathcal{M}) \longrightarrow \operatorname{Ex}_\Sigma(a/S, \mathcal{M}''). \end{aligned}$$

PROOF. By the homogeneity of p there are fibred sums in $\mathbf{Ex}_\Sigma(a)$ and $\mathbf{Ex}_\Sigma(a/S)$, or, equivalently, fibred products in the associated opposite categories. Moreover $\mathbf{Ex}_\Sigma(a)(0)$ and $\mathbf{Ex}_\Sigma(a)(0)$ are just the trivial categories with the only object a. Thus the result follows from 3.2.2. □

The next result is just a reformulation of 3.2.8. It is a basic exact sequence in deformation theory and will play an important role in the following.

Theorem 3.3.4. *The sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{Aut}_\Sigma(a/S,\mathcal{M}) \longrightarrow \mathrm{Aut}_\Sigma(a,\mathcal{M}) \longrightarrow \mathrm{Der}_\Sigma(\mathcal{O}_S,\mathcal{M}) \\ \xrightarrow{\delta_{KS}} \mathrm{Ex}_\Sigma(a/S,\mathcal{M}) \longrightarrow \mathrm{Ex}_\Sigma(a,\mathcal{M}) \longrightarrow \mathrm{Ex}_\Sigma(S,\mathcal{M}) \end{aligned}$$

is natural in $\mathcal{M} \in \mathbf{Coh}(S)$ *and exact.*

The connecting homomorphism δ_{KS} in the exact sequence above will be called the *Kodaira Spencer map.* Because of its importance we give its explicit description in terms of extensions.

3.3.5. Let $\vartheta : \mathcal{O}_S \to \mathcal{M}$ be a Σ-derivation and $1-\vartheta : \mathcal{O}_S \to \mathcal{O}_S[\mathcal{M}]$ be the associated $\mathcal{O}_\Sigma$-algebra homomorphism. This gives a morphism again denoted by $1-\vartheta : S[\mathcal{M}] \to S$ retracting the inclusion $S \hookrightarrow S[\mathcal{M}]$. We set $a_\vartheta := a \times_S S[\mathcal{M}]$, i.e.

$$\begin{tikzcd} a_\vartheta \arrow[r] \arrow[d, mapsto] & a \arrow[d, mapsto] \\ S[\mathcal{M}] \arrow[r, "1-\vartheta"] & S \end{tikzcd}$$

is cartesian. By the construction of sect. 2.2. we have

$$\delta_{KS}(\vartheta) = [a_\vartheta] \in \mathrm{Ex}_\Sigma(a/S,\mathcal{M}) \ .$$

Another useful fact of these constructions is the compatibility with finite maps. We know from 2.4.6 that for a finite morphism of Σ-spaces $f : S \to T$ there are functorial maps

$$\begin{aligned} f_* : \mathrm{Ex}_\Sigma(S,\mathcal{M}) &\longrightarrow \mathrm{Ex}_\Sigma(T, f_*(\mathcal{M})) \\ f_* : \mathrm{Der}_\Sigma(S,\mathcal{M}) &\longrightarrow \mathrm{Der}_\Sigma(T, f_*(\mathcal{M}). \end{aligned}$$

These maps generalize to arbitrary deformation theories:

3.3.6. Let $\tilde{f} : a \to b$ be a morphism in $\mathbf{F}$ over $f : S \to T$ such that f is a finite morphism of complex spaces. Then $\tilde{f}$ induces a functor

$$\tilde{f}_* : \mathbf{Ex}_\Sigma(a) \to \mathbf{Ex}_\Sigma(b)$$

via $a' \mapsto b' := b \amalg_a a'$. If a' is an extension of a by the coherent $\mathcal{O}_S$-module $\mathcal{M}$ then b' is an extension of b by $f_*(\mathcal{M})$. It is easily seen from the associativity of coproducts that $\tilde{f}_*$ commutes with fibred coproducts. Hence there are induced maps also denoted by $\tilde{f}_*$

$$\begin{aligned} \tilde{f}_* : \mathrm{Ex}_\Sigma(a,\mathcal{M}) &\longrightarrow \mathrm{Ex}_\Sigma(b, f_*(\mathcal{M})) \\ \tilde{f}_* : \mathrm{Aut}_\Sigma(a,\mathcal{M}) &\longrightarrow \mathrm{Aut}_\Sigma(b, f_*(\mathcal{M}). \end{aligned}$$

They are functorial in $\mathcal{M}$ and therefore are $\Gamma(T,\mathcal{O}_T)$-linear, see 2.4.13 (2). In a similar way $\tilde{f}$ gives maps

$$\begin{aligned} \tilde{f}_* : \mathrm{Ex}_\Sigma(a/S,\mathcal{M}) &\longrightarrow \mathrm{Ex}_\Sigma(b/T, f_*(\mathcal{M})) \\ \tilde{f}_* : \mathrm{Aut}_\Sigma(a/S,\mathcal{M}) &\longrightarrow \mathrm{Aut}_\Sigma(b/T, f_*(\mathcal{M}). \end{aligned}$$

which by the same reason as above are $\Gamma(T,\mathcal{O}_T)$-linear. *These last two maps are even isomorphisms*. In fact, if $f[\mathcal{M}] : S[\mathcal{M}] \to T[f_*(\mathcal{M})]$ is the map induced by f

then taking the pullback gives maps

$$\begin{aligned} f[\mathcal{M}]^* &: \operatorname{Ex}_\Sigma(b/T, f_*(\mathcal{M})) \longrightarrow \operatorname{Ex}_\Sigma(a/S, \mathcal{M}) \\ f[\mathcal{M}]^* &: \operatorname{Aut}_\Sigma(b/T, f_*(\mathcal{M})) \longrightarrow \operatorname{Aut}_\Sigma(a/S, \mathcal{M}) \end{aligned}$$

which are inverse to $\tilde{f}_*$.

These constructions are compatible with exact sequences in $\mathcal{M}$, se 3.3.3, as well with the Kodaira-Spencer sequence in 3.3.4. Moreover, if $\tilde{g} : b \to c$ is a further morphism then $\tilde{g}_* \circ \tilde{f}_* = (\tilde{g}\tilde{f})_*$.

As an example we compute the Kodaira-Spencer class of deformations of complex spaces.

PROPOSITION 3.3.7. *Let $f : X \to S$ be a flat morphism of complex spaces which defines an object f of the groupoid defined in example 3.1.6 (1). Then for* $\mathcal{M} \in \mathbf{Coh}(S)$

(1) $\operatorname{Ex}_\Sigma(f/S, \mathcal{M}) \cong \operatorname{Ex}_S(X, f^*(\mathcal{M}))$.

(2) $\operatorname{Aut}_\Sigma(f/S, \mathcal{M}) \cong \operatorname{Der}_S(\mathcal{O}_X, f^*(\mathcal{M}))$.

(3) $\operatorname{Aut}_\Sigma(f, \mathcal{M})$ *is the set of all compatible derivations in*

$$\operatorname{Der}_\Sigma(\mathcal{O}_S, \mathcal{M}) \times \operatorname{Der}_\Sigma(\mathcal{O}_X, f^*\mathcal{M}).$$

PROOF. By definition an element of $\operatorname{Ex}_\Sigma(f/S, \mathcal{M})$ is given by a flat map $f' : X' \to S[\mathcal{M}]$ which induces f over $S \hookrightarrow S[\mathcal{M}]$. As f' is flat the inverse image of the exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{O}_{S[\mathcal{M}]} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

gives an exact sequence on X'

$$0 \longrightarrow f^*(\mathcal{M}) \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Thus X' is an S-extension of X by $f^*(\mathcal{M})$. Conversely, if X' is an S-extension of X by $f^*(\mathcal{M})$ then X' may be considered as a space over $S[\mathcal{M}]$ in a natural way. By the following lemma X' is $S[\mathcal{M}]$-flat which proves (1). The proofs of (2) and (3) are easy consequences of 2.3.10 and left to the reader. □

LEMMA 3.3.8. *Let A be a ring and $I \subseteq A$ be a nilpotent ideal. Then for any A-module M the following are equivalent:*

(1) *M is A-flat.*

(2) *M/IM is A/I-flat, and the natural map $I \otimes M \to M$ is injective.*

For a *proof*, see e.g. [**Mat**] (22.3), (1) ⇔ (3).

We will show how to compute the Kodaira-Spencer map for deformations $f : X \to S$ of complex spaces which can be embedded in a diagram

$$\begin{tikzcd} X \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & Y := U \times S \arrow[dl, "proj"] \\ & S , & \end{tikzcd}$$

where i is a closed embedding and U is some complex space. Recall that in this case there is a canonical map $\pi : \operatorname{Hom}(\mathcal{J}/\mathcal{J}^2, f^*\mathcal{M}) \to \operatorname{Ex}_S(X, f^*\mathcal{M})$, where $\mathcal{J} \subseteq \mathcal{O}_Y$ is the ideal sheaf of X in Y, see 2.5.1. Identifying $\operatorname{Ex}_S(X, f^*\mathcal{M})$ with $\operatorname{Ex}_\Sigma(f/S, \mathcal{M})$ by the preceding proposition we will give a more explicit description of the Kodaira Spencer map

$$\delta_{KS} : \operatorname{Der}_\Sigma(\mathcal{O}_S, \mathcal{M}) \longrightarrow \operatorname{Ex}_S(X, f^*\mathcal{M}).$$

First we remark that by the product structure of $Y = U \times S$ there is a canonical map $\mathrm{Der}_\Sigma(\mathcal{O}_S, \mathcal{M}) \to \mathrm{Der}_\Sigma(\mathcal{O}_Y, p_2^*\mathcal{M})$. Composing furthermore with the surjection $p_2^*\mathcal{M} \to f^*\mathcal{M}$ gives a map denoted by $\lambda_f : \mathrm{Der}_\Sigma(\mathcal{O}_S, \mathcal{M}) \to \mathrm{Der}_\Sigma(\mathcal{O}_Y, f^*\mathcal{M})$. With these notations we get the following description.

Proposition 3.3.9. *For $\mathcal{M} \in \mathbf{Coh}(S)$ the diagram*

$$\begin{tikzcd}
\mathrm{Der}_\Sigma(\mathcal{O}_S, \mathcal{M}) \arrow[r, "\delta_{KS}"] \arrow[d, "\lambda_f"'] & \mathrm{Ex}_S(X, f^*\mathcal{M}) \\
\mathrm{Der}_\Sigma(\mathcal{O}_Y, f^*\mathcal{M}) \arrow[r, "\pi"] & \mathrm{Hom}(\mathcal{J}/\mathcal{J}^2, f^*\mathcal{M}) \arrow[u, "can"']
\end{tikzcd}$$

commutes.

Proof. Let $\vartheta \in \mathrm{Der}_\Sigma(\mathcal{O}_S, \mathcal{M})$ be a Σ-derivation. By definition of the Kodaira-Spencer map $\delta_{K/S}(\vartheta)$ is represented by X_ϑ where X_ϑ is the fibre product in the first of the diagrams

$$\begin{tikzcd}
X_\vartheta \arrow[r] \arrow[d] & X \arrow[d] & & X_\vartheta \arrow[r] \arrow[d] & X \arrow[d] \\
S[\mathcal{M}] \arrow[r, "1-\vartheta"] & S & & Y[f^*\mathcal{M}] \arrow[r, "1-\lambda_f(\vartheta)"] & Y.
\end{tikzcd}$$

This implies that the second of these diagrams is cartesian too. Now the claim follows from 2.5.4. □

Another case where we can make the Kodaira Spencer map explicit is the case of deformations $f : X \to S$ which are smooth maps. By 2.4.12

$$(*) \qquad \mathrm{Ex}_S(X, f^*\mathcal{M}) \xrightarrow{\cong} \mathrm{Ext}^1(\Omega^1_{X/S}, f^*\mathcal{M}) \cong H^1(X, \Theta_{X/S} \otimes f^*\mathcal{M}),$$

for $\mathcal{M} \in \mathbf{Coh}(S)$. Since f is smooth, the sequence

$$0 \longrightarrow f^*(\Omega^1_{S/\Sigma}) \longrightarrow \Omega^1_{X/\Sigma} \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

is exact. The boundary homomorphism in the associated Ext-sequence gives a map

$$\delta : \mathrm{Hom}(f^*(\Omega^1_{S/\Sigma}), f^*\mathcal{M}) \longrightarrow \mathrm{Ext}^1(\Omega^1_{X/S}, f^*\mathcal{M}).$$

Proposition 3.3.10. *The diagram*

$$\begin{tikzcd}
\mathrm{Der}_\Sigma(\mathcal{O}_S, \mathcal{M}) = \mathrm{Hom}(\Omega^1_{S/\Sigma}, \mathcal{M}) \arrow[r, "\delta_{K/S}"] \arrow[d, "f^*"] & \mathrm{Ex}_S(X, \mathcal{M}) \arrow[d, "\cong"'] \\
\mathrm{Hom}(f^*\Omega^1_{S/\Sigma}, f^*\mathcal{M}) \arrow[r, "\delta"'] & \mathrm{Ext}^1(\Omega^1_{X/S}, f^*\mathcal{M})
\end{tikzcd}$$

commutes. In particular, $\mathrm{Ex}_S(X, \mathcal{M}) \cong H^1(X, \Theta_{X/S} \otimes f^*\mathcal{M})$.

Proof. Given $\vartheta \in \mathrm{Der}_\Sigma(\mathcal{O}_S, \mathcal{M})$ let X_ϑ be as above, which is an extension of X by $f^*(\mathcal{M})$. By definition, see 2.4.12, under the identification $(*)$ the extension X_ϑ is identified with the extension in $\mathrm{Ext}^1(\Omega^1_{X/S}, f^*\mathcal{M})$ represented by the top lines in the diagram

$$\begin{tikzcd}
0 \arrow[r] & f^*\mathcal{M} \arrow[r] & \Omega^1_{X_\vartheta/S} \otimes_{\mathcal{O}_{X_\vartheta}} \mathcal{O}_X \arrow[r] & \Omega^1_{X/S} \arrow[r] & 0 \\
0 \arrow[r] & f^*\Omega^1_S \arrow[r] \arrow[u, "-f^*\vartheta"] & \Omega^1_X \arrow[r] \arrow[u, "dp"'] & \Omega^1_{X/S} \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $p : X_\vartheta \to X$ is the projection. The commutativity of this diagram is exactly the statement of the proposition as the pushout of the bottom line along $-f^*\vartheta$ represents the image of $f^*\vartheta$ under the boundary homomorphism δ in terms of extensions, see [**Bou**]. □

3.4. Formally Versal Deformations

In this section we will introduce the basic concepts of versal and formally versal deformations. In order to formulate this it is convenient to work with germs instead of globally defined objects.

Let $\mathbf{An}_{(\Sigma,0)}$ denote the category of germs $(S,0)$ of complex spaces over a given germ $(\Sigma,0)$. For simplicity, the base point of a germ will almost always be denoted by 0.

Definition 3.4.1. A fibration in groupoids $p : \mathbf{F} \to \mathbf{An}_{(\Sigma,0)}$ is called a *(local) deformation theory* if the condition (H) of 3.1.11 is satisfied for germs $(S,0)$, $(S',0)$, $(T,0)$.

If a is an element in $\mathbf{An}_{(\Sigma,0)}(S,0)$ which induces a_0 on the (simple) point 0 then we call a a *deformation* of a_0. As a basic example we treat deformations of singularities.

Example 3.4.2 (Deformations of singularities). We consider the following (local) deformation theory $p : \mathbf{F} \to \mathbf{An}_{(\Sigma,0)}$. An object in $\mathbf{F}$ over $(S,0) \in \mathbf{An}_{(\Sigma,0)}$ is a germ $(X,0) \in \mathbf{An}_{(\Sigma,0)}$ such that the structure morphism

$$f : (X,0) \longrightarrow (S,0)$$

is flat. One interprets such a morphism as a deformation of the special fibre $(X_0,0) := \big(f^{-1}(0),0\big)$. We will call $(X,0)$ the (germ of the) *total space* and $(S,0)$ the *basis of the deformation*. If $(X',x0) \in \mathbf{An}_{(S',0)}$ is another flat germ then a morphism from $(X',0)$ to $(X,0)$ of deformations is a commutative cartesian diagram

$$\begin{tikzcd}
(X',0) \arrow[r] \arrow[d] & (X,0) \arrow[d] \\
(S',0) \arrow[r] & (S,0)
\end{tikzcd}$$

where the vertical arrows are the structure maps. Observe that in this case the special fibers of the deformations are equal.

It follows from 2.4.4 that p is indeed a deformation theory.

3.4.3. Let $p : \mathbf{F} \to \mathbf{An}_\Sigma$ be a (global) deformation theory and $0 \in \Sigma$ a fixed point. One can associate to p in a natural way a local deformation theory $p_0 : \mathbf{F}_0 \to \mathbf{An}_{(\Sigma,0)}$ by taking

$$\mathbf{F}_0(S,0) = \varinjlim{}_U \mathbf{F}(U),$$

where U runs through the open neighbourhoods of 0 in S. In other words, an object of $\mathbf{F}_0$ over $(S,0)$ is an object $a \in \mathbf{F}(U)$ which is defined on some open neighbourhood U of 0 in S, and two objects $a \in \mathbf{F}(U)$ and $b \in \mathbf{F}(V)$ are considered to be equal in $\mathbf{F}(S,0)$ if their restrictions to a suitable open neighbourhood $W \subseteq U \cap V$ are equal. The morphisms in $\mathbf{F}_0$ are defined in an obvious way. To every object $a \in \mathbf{F}(S)$ and a point $0 \in S$ we can associate its so called germ $(a,0)$ in $\mathbf{F}(S,0)$.

In order to introduce (formal) versality it is necessary to extend the deformations to include also formal objects.

3.4.4. Let $p : \mathbf{F} \to \mathbf{An}_{(\Sigma,0)}$ be a deformation theory. Let $\widehat{\mathbf{An}}_{(\Sigma,0)}$ be the category of all formal germs of complex spaces, i.e. all germs $\bar{S} = (0, \mathcal{O}_{\bar{S}})$ where $\mathcal{O}_{\bar{S}}$ is a local Noetherian $\mathcal{O}_{\Sigma,0}$-algebra with residue field $\mathbb{C}$, which is complete with respect to its maximal ideal. Thus $\widehat{\mathbf{An}}^{opp}_{(\Sigma,0)}$ is just the category of local complete analytic $\mathbb{C}$-algebras that are $\mathcal{O}_{\Sigma,0}$-algebras. For $\bar{S} \in \widehat{\mathbf{An}}_{(\Sigma,0)}$ we denote by $\bar{S}_n$ the n-th infinitesimal neighbourhood, i.e. $\bar{S}_n$ is the fat point

$$\bar{S}_n = \left(0, \mathcal{O}_{\bar{S}}/\mathfrak{m}^{n+1}_{\bar{S},0}\right).$$

We can associate to p a so called *formal deformation theory*

$$\hat{p} : \hat{\mathbf{F}} \longrightarrow \widehat{\mathbf{An}}_{(\Sigma,0)}$$

in the following way. Let $\bar{S} \in \widehat{\mathbf{An}}_{(\Sigma,0)}$ be a formal germ of a complex space, with infinitesimal neighbourhoods $\bar{S}_n \in \mathbf{An}_\Sigma$. Then an object $\bar{a} \in \hat{\mathbf{F}}(\bar{S})$ is a sequence of morphisms

$$a_0 \hookrightarrow a_1 \hookrightarrow \cdots \hookrightarrow a_n \hookrightarrow a_{n+1} \hookrightarrow \cdots$$

with $a_n \hookrightarrow a_{n+1}$ in $\mathbf{F}$ lying over $\bar{S}_n \hookrightarrow \bar{S}_{n+1}$. We write shortly $\bar{a} = (a_n)$ in this case and call $\bar{a}$ a *formal deformation* of a_0. A morphism $\bar{a} \to \bar{b}$ in $\hat{F}$ over $\bar{S} \to \bar{T}$ is a chain of morphisms $a_n \to b_n$ over $\bar{S}_n \to \bar{T}_n$ which are compatible with the transition maps $a_n \hookrightarrow a_{n+1}$ and $b_n \hookrightarrow b_{n+1}$. Obviously $\hat{p}$ is again a fibration in groupoids.

3.4.5. If $(S, 0) \in \mathbf{An}_{(\Sigma,0)}$ then we can consider the completion $\hat{S} = (0, \hat{\mathcal{O}}_{S,0})$, where $\hat{\mathcal{O}}_{S,0}$ is the $\mathfrak{m}_{S,0}$-adic completion of $\mathcal{O}_{S,0}$. Obviously then $\hat{S} \in \widehat{\mathbf{An}}_{(\Sigma,0)}$. To every object $a \in \mathbf{F}(S, 0)$ we can associate the formal object $\hat{a} \in \mathbf{F}(\hat{S})$ given by $\hat{a}_n = a \times_S \hat{S}_n$. We will call $\hat{a}$ the *formal completion* of a. Thus we obtain a commutative diagram

$$\begin{tikzcd}
\mathbf{F} \arrow[r] \arrow[d] & \mathbf{An}_{(\Sigma,0)} \arrow[d] \\
\hat{\mathbf{F}} \arrow[r] & \widehat{\mathbf{An}}_{(\Sigma,0)}
\end{tikzcd}$$

where the vertical maps are given by the completion.

Observe that for a fat point S over Σ one has $\hat{\mathbf{F}}(S) \cong \mathbf{F}(S, 0)$ in a canonical way. We will now introduce versality and formal versality. In the following, we will call an extension of germs of (formal) complex spaces $(T, 0) \hookrightarrow (T', 0)$ a *small extension* if it is an extension by a module $\mathcal{M}$ of length one, i.e. $\mathcal{M} \cong \mathbb{C}$. Similarly an extension in $\mathbf{F}$ or $\hat{\mathbf{F}}$ is called small if the underlying extension of (formal) germs of complex spaces is small.

Definition 3.4.6. A formal deformation $\bar{a} \in \mathbf{F}(\bar{S})$ of $a_0 \in \mathbf{F}(\{0\})$ is called *formally versal* if the following lifting property is satisfied.

FV: For every diagram of solid arrows

$$\begin{tikzcd}
b \arrow[dd, hook] \arrow[dr] & \\
& \bar{a} \\
b' \arrow[ur, dotted] &
\end{tikzcd}$$

where $p(b) \to p(b')$ is a small extension of fat points, there is a lifting as indicated by the dotted arrow.

Similarly, if $a \in \mathbf{F}(S, 0)$ then a is called formally versal if the lifting property above is satisfied for a instead of $\bar{a}$.

We remark that a is formally versal iff the completion $\hat{a} \in F(\hat{S})$ is formally versal. This follows immediately from the fact that the morphisms $T \to S$ from fat points T into S are in a 1-1-correspondence with the morphism $T \to \hat{S}$.

The property (FV) implies the following stronger lifting property.

Lemma 3.4.7. *Let $\bar{a}$ be a formally versal deformation of a_0. Then the lifting property (FV) is satisfied for any morphism $\bar{b} \to \bar{b}'$ in $\hat{\mathbf{F}}$ that lies over a closed embedding $\bar{T} \hookrightarrow \bar{T}'$.*

Proof. The lifting property is obviously satisfied if $\bar{T} \hookrightarrow \bar{T}'$ is an embedding of fat points as follows by an easy induction from (FV). In the general case we will construct inductively compatible morphisms $f'_n : b'_n \to \bar{a}$ lifting the morphisms $f_n : b_n \to \bar{a}$ induced by $\bar{f}$. For $n = 0$ there is nothing to do. Assume now that $f'_{n-1} (n \geq 1)$ has already been constructed. Then the canonical map

$$c_n := b_n \amalg_{b_{n-1}} b'_{n-1} \longrightarrow b'_n$$

is lying over a closed embedding of fat points. Hence there exists a morphism $f'_n : b'_n \to \bar{a}$ lifting the morphism $f_n \amalg f'_{n-1} : c_n \to \bar{a}$. Clearly f'_n lifts f'_{n-1} as desired. □

Definition 3.4.8. Let $(S, 0)$ be a germ of complex space over $(\Sigma, 0)$. A deformation $a \in F(S, s_0)$ of a_0 is called *versal* if the lifting property (FV) for a instead of $\bar{a}$ is satisfied for any morphism of local deformation $b \hookrightarrow b'$ in F_0 such that $(T, 0) := p(b) \hookrightarrow (T', 0) := p(b')$ is a closed embedding, i.e. a diagram

$$\begin{tikzcd} b \arrow[d, hook] \arrow[dr] & \\ b' \arrow[r, dotted] & a \end{tikzcd}$$

with $b \in F(T, 0)$, $b' \in F(T', 0)$ can be completed as indicated by the dotted arrow.

Remark 3.4.9. Let $p : \mathbf{F} \to \mathbf{An}_\Sigma$ be a global deformation theory and $a \in \mathbf{F}(S)$. If 0 is a point of S then we can associate to $\mathbf{F}$ a local deformation theory and to a its germ, see 3.4.3. We call a versal resp. formally versal at 0 if the germ of a at 0 has the corresponding property.

Remarks 3.4.10. (1) Assume that $\bar{a} \in \hat{F}(\bar{S})$ is a formal deformation of a_0 and let $f : \bar{S}' \to \bar{S}$ be a smooth map of formal germs, i.e. $\mathcal{O}_{\bar{S}'} \cong \mathcal{O}_{\bar{S}}[\![T_1, \ldots, T_n]\!]$. Then $\bar{a}$ is formally versal iff $f^*(\bar{a}) \in \hat{F}(\bar{S}')$ is formally versal. This follows from the definitions and the fact that for a fat point T and $b \in \hat{F}(T)$ we have

$$\begin{aligned} \mathrm{Mor}(b, f^*\bar{a}) &= \mathrm{Mor}(b, \bar{a}) \times_{\mathrm{Mor}(T,S)} \mathrm{Mor}(T, \bar{S}') \\ &\cong \mathrm{Mor}(b, \bar{a}) \times \mathfrak{m}_T^{\oplus n}, \end{aligned}$$

since the liftings of morphisms $T \to \bar{S}$ to morphisms $T \to \bar{S}'$ are determined by the images of the indeterminates $T_i \in \mathcal{O}_{\bar{S}'}$ in $\mathfrak{m}_T$, the maximal ideal of $\mathcal{O}_T$, and conversely, every tuple in $\mathfrak{m}_T^{\oplus n}$ defines a lifting of a morphism $T \to \bar{S}$ to a morphism $T \to \bar{S}'$.

(2) Similarly as above, if $a \in F(S,0)$ and $p : (S',0) \to (S,0)$ is a smooth map then a is versal iff $p^*(a)$ is versal. This is easily seen with the same arguments as above.

We will see later that the converse of these remarks is also true, i.e. that two formally versal resp. versal deformations differ by a smooth factor, see 3.5.8.

LEMMA 3.4.11. *Let $\bar S \to \bar S'$ is an extension of formal germs of complex spaces by a $\mathcal{O}_{\bar S}$-module $\bar{\mathcal{M}}$ and $\bar S \to \bar T$ a finite map. Set $\bar T' := \bar T \amalg_{\bar S} \bar S'$. Let the index n indicate the n-th infinitesimal neighbourhoods. Then $\bar T'_n = \bar T_n \amalg_{\bar S_n} \bar S'_n$.*

PROOF. Let $A = \mathcal{O}_{\bar S}$, $A' = \mathcal{O}_{\bar S'}$, $B = \mathcal{O}_{\bar T}$, $B' = \mathcal{O}_{\bar T'}$, be the associated complete local rings. Then $B' \cong A' \times_A B$ and $\mathfrak{m}_{B'} = \mathfrak{m}_{A'} \times_A \mathfrak{m}_B$. Hence $\mathfrak{m}_{B'}^{n+1} = \mathfrak{m}_{A'}^{n+1} \times_A \mathfrak{m}_B^{n+1}$ and so

$$B'_n = B'/\mathfrak{m}_{B'}^{n+1} \cong A'/\mathfrak{m}_{A'}^{n+1} \times_A B/\mathfrak{m}_B^{n+1} \cong A'_n \times_{A_n} B_n$$

proving the lemma. □

COROLLARY 3.4.12. *A formal deformation theory $\hat p : \hat F \to \widehat{\mathbf{An}}_{(\Sigma,0)}$ satisfies the homogeneity condition, i.e. if a diagram*

$$\begin{tikzcd} \bar a \arrow[r] \arrow[d] & \bar a & \textit{over} & \bar S \arrow[r] \arrow[d] & \bar S' \\ \bar b & & & \bar T & \end{tikzcd}$$

is given, where $\bar S \to \bar S'$ is an extension and $\bar S \to \bar T$ is finite, then the fibred sum $\bar b' = \bar b \coprod_{\bar a} \bar a'$ exists.

PROOF. Let $\bar T'$ be the fibred sum $\bar T \amalg_{\bar S} \bar S'$. By the homogeneity of $p : F \to \mathbf{An}_\Sigma$, the fibred sum $b'_n := b_n \coprod_{a_n} a'_n$ exists and yields an object over $\bar T_n \coprod_{S_n} \bar S'_n$, which is just T'_n by 3.4.11. Then $\bar b' = (b'_n)$ gives an object in $\hat F(\bar T')$ which is easily seen to be a fibred sum. □

3.4.13. Let $\bar S \in \widehat{\mathbf{An}}_{(\Sigma,0)}$ and $\bar a \in \hat F(\bar S)$. Then we can form as in 3.3.1 the category of extension $\mathbf{Ex}_{(\Sigma,0)}(\bar a)$ which is cofibred over $\mathbf{Coh}(\bar S)$, i.e. an object in $\mathbf{Ex}_{(\Sigma,0)}(\bar a)$ over a coherent $\mathcal{O}_{\bar S}$-module $\bar{\mathcal{M}}$ is given by a pair $(\bar a \hookrightarrow \bar a', u)$ such that $\bar S = p(\bar a) \hookrightarrow p(\bar a') = \bar S'$ is an extension of $\bar S$ by $\bar{\mathcal{M}}$. As in 3.3.1 and 3.3.2 we denote the set of isomorphism classes of the fibre $\mathbf{Ex}_{(\Sigma,0)}(\bar a)(\bar{\mathcal{M}})$ by $\mathrm{Ex}_{(\Sigma,0)}(\bar a, \bar{\mathcal{M}})$ and the set of automorphisms of the trivial extension $\bar a[\bar{\mathcal{M}}]$ by $\mathrm{Aut}_{(\Sigma,0)}(\bar a, \bar{\mathcal{M}})$. Similarly, we can form the sets $\mathrm{Ex}_{(\Sigma,0)}(\bar a/\bar S, \bar{\mathcal{M}})$ and $\mathrm{Aut}_{(\Sigma,0)}(\bar a/\bar S, \bar{\mathcal{M}})$, see 3.3.1 and 3.3.2. The homogeneity of $\hat p : \hat F \to \widehat{\mathbf{An}}_{(\Sigma,0)}$ implies as in 3.3.3 that there are fibred products in $\mathbf{Ex}_{(\Sigma,0)}(\bar a)$ and that

$$\begin{array}{rcl} \mathrm{Aut}_{(\Sigma,0)}(\bar a, \bar{\mathcal{M}}) & , & \mathrm{Aut}_{(\Sigma,0)}(\bar a/\bar S, \bar{\mathcal{M}}) \\ \mathrm{Ex}_{(\Sigma,0)}(\bar a, \bar{\mathcal{M}}) & , & \mathrm{Ex}_{(\Sigma,0)}(\bar a/\bar S, \bar{\mathcal{M}}) \end{array}$$

carry natural $\mathcal{O}_{\bar S}$-module structures. Moreover the results of sect. 2.3 also hold m.m. which we will use in the following without any comment. In particular, a short exact seqence of $\mathcal{O}_{\bar S}$-modules induces exact sequences as in 3.3.3 (3), and there is a Kodaira-Spencer sequence as in 3.3.4. Taking completion gives a natural functor $\mathbf{Ex}_{(\Sigma,0)}(a) \to \mathbf{Ex}_{(\Sigma,0)}(\hat{a})$ if $a \in \mathbf{F}(S,0)$ is a convergent object. In particular, for a coherent $\mathcal{O}_S$-module $\mathcal{M}$ there are natural maps of $\mathcal{O}_{S,0}$ modules

$$\begin{aligned} \mathrm{Aut}_{(\Sigma,0)}(a/S,\mathcal{M}) &\longrightarrow \mathrm{Ex}_{(\Sigma,0)}(\hat{a}/\hat{S},\hat{\mathcal{M}}) \\ \mathrm{Aut}_{(\Sigma,0)}(a,\mathcal{M}) &\longrightarrow \mathrm{Ex}_{(\Sigma,0)}(\hat{a},\hat{\mathcal{M}}) \\ \mathrm{Ex}_{(\Sigma,0)}(a/S,\mathcal{M}) &\longrightarrow \mathrm{Ex}_{(\Sigma,0)}(\hat{a}/\hat{S},\hat{\mathcal{M}}) \\ \mathrm{Ex}_{(\Sigma,0)}(a,\mathcal{M}) &\longrightarrow \mathrm{Ex}_{(\Sigma,0)}(\hat{a},\hat{\mathcal{M}}) \end{aligned}$$

which are functorial with respect to exact sequences in $\mathcal{M}$ and compatible with the Kodaira-Spencer sequence.

For Artinian modules the Ex-groups do not change under completion. More precisely, the following lemma holds.

LEMMA 3.4.14. *Let $\mathcal{M}$ be an Artinian $\mathcal{O}_{\bar{S}}$-module with $\mathfrak{m}_{\bar{S}}^{k+1}\mathcal{M} = 0$ and $\bar{a} \in$ $\mathbf{Ex}(\bar{S},0)$. Then the following hold.*

(1) *The natural maps of $\mathcal{O}_{\bar{S}}$-modules*

(a) $$\varinjlim_{n \geq k} \mathrm{Ex}_{(\Sigma,0)}(a_n,\mathcal{M}) \longrightarrow \mathrm{Ex}_{(\Sigma,0)}(\bar{a},\mathcal{M})$$

(b) $$\mathrm{Ex}_{(\Sigma,0)}(a_k/S_k,\mathcal{M}) \longrightarrow \mathrm{Ex}_{(\Sigma,0)}(\bar{a}/\bar{S},\mathcal{M})$$

are bijective.

(2) *If $\bar{a} = \hat{a}$ for some $a \in \mathbf{F}$ lying over the convegent germ $(S,0)$ then the natural maps*

(c) $$\varinjlim_{n \geq k} \mathrm{Ex}_{(\Sigma,0)}(a_n,\mathcal{M}) \longrightarrow \mathrm{Ex}_{(\Sigma,0)}(a,\mathcal{M}) \longrightarrow \mathrm{Ex}_{(\Sigma,0)}(\bar{a},\mathcal{M})$$

(d) $$\mathrm{Ex}_{(\Sigma,0)}(a_k/S_k,\mathcal{M}) \longrightarrow \mathrm{Ex}_{(\Sigma,0)}(a/S,\mathcal{M}) \longrightarrow \mathrm{Ex}_{(\Sigma,0)}(\bar{a}/\bar{S},\mathcal{M})$$

are bijective.

PROOF. For the proof of (a) let $[\bar{b}] \in \mathrm{Ex}_{(\Sigma,0)}(\bar{a},\mathcal{M})$ be an element represented by the extension $\bar{b} = (\bar{a} \hookrightarrow \bar{b}, \bar{u})$. Consider the exact sequence

$$0 \longrightarrow \mathcal{M} \xrightarrow{\bar{u}} \mathcal{O}_{\bar{T}} \longrightarrow \mathcal{O}_{\bar{S}} \longrightarrow 0.$$

By the lemma of Artin-Rees and the assumption that $\mathcal{M}$ is Artinian we get that $\mathfrak{m}_{\bar{T}}^n \cap \bar{u}(\mathcal{M}) = 0$ for $n \gg 0$. Denoting by the index n the n-th infinitesimal neighbourhood this implies that for $n \geq m \gg 0$ the diagrams

$$\begin{tikzcd} a_m \arrow[r] \arrow[d] & a_n \arrow[d] \\ b_m \arrow[r] & b_n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S_m \arrow[r] \arrow[d] & S_n \arrow[d] \\ T_m \arrow[r] & T_n \end{tikzcd}$$

are cocartesian and so $\bar{b}$ is already uniquely determined by b_m. In the case of (b) we have $\mathcal{O}_{\bar{T}} \cong \mathcal{O}_{\bar{T}} \times \mathcal{M}$ and so $\mathfrak{m}_{\bar{T}}^{k+1} \cap \bar{u}(\mathcal{M}) = 0$. Thus the above diagram is cocartesian for $n \geq m := k$, and the bijectivity follows as before.

With the same argument we obtain that the first map in (c) resp. (d) is bijective. As the composition with the second map is bijective by (1) the result follows. □

In the following proposition we give a useful criterion for an object $\bar{a} \in \hat{F}(\bar{S})$ to be formally versal.

Proposition 3.4.15. *The following are equivalent.*
(1) $\bar{a}$ *is formally versal.*
(2) $\mathrm{Ex}_{(\Sigma,0)}(\bar{a},\mathbb{C}) = 0$.
(3) $\mathrm{Ex}_{(\Sigma,0)}(\bar{a},\bar{\mathcal{M}}) = 0$ *for every finite* $\mathcal{O}_{\bar{S}}$*-module* $\bar{\mathcal{M}}$.

Proof. For the proof of (1)⇒(3) consider $[\bar{a}'] \in \mathrm{Ex}_{(\Sigma,0)}(\bar{a},\bar{\mathcal{M}})$, i.e. $\alpha : \bar{a} \hookrightarrow \bar{a}'$ is an extension of $\bar{a}$ by $\bar{\mathcal{M}}$. By the formal versality, see 3.4.7, there exists a morphism $\beta : \bar{a}' \to \bar{a}$ with $\beta \circ \alpha = id_{\bar{a}}$. This shows that $[\bar{a}'] = 0$ in $\mathrm{Ex}_{(\Sigma,0)}(\bar{a},\bar{\mathcal{M}})$.

(3)⇒(2) is trivial. Finally, assume that (2) holds and consider a diagram of solid arrows

$$\begin{tikzcd} b \arrow[r, hook] \arrow[d, "f"'] & b' \\ \bar{a} & \end{tikzcd}$$

with $p(b) \hookrightarrow p(b')$ a small extension of fat points by $\mathbb{C}$. The fibred sum $\bar{a}' = \bar{a} \coprod_b b'$ then defines a small extension of $\bar{a}$ and so an element of $\mathrm{Ex}_{(\Sigma,0)}(\bar{a},\mathbb{C})$, which by assumption is zero. Thus there exists an arrow $\bar{a}' \to \bar{a}$ retracting the inclusion $\bar{a} \hookrightarrow \bar{a}'$. Composing this retraction with the natural map $b' \to \bar{a}'$ gives a lifting of f. □

This immediately implies the following simple criterion for formal versality which will be useful in proving openness of versality in sect. ??.

Corollary 3.4.16. *Let* $p : \mathbf{F} \to \mathbf{An}_\Sigma$ *be a (global) deformation theory and* $a \in F(S)$. *Then* a *is formally versal in* $s_0 \in S$ *iff* $\mathrm{Ex}_\Sigma(a,\mathbb{C}_{s_o}) = 0$ *where* $\mathbb{C}_{s_0}$ *denotes in brief the sheaf* $\mathcal{O}_S/\mathfrak{m}_{S,s_0}$.

In the rest of this section, consider again a local deformation theory $p : \mathbf{F} \to \mathbf{An}_{(\Sigma,0)}$.

Corollary 3.4.17. *Let* $\bar{a} \in \mathbf{F}(\bar{S})$ *be formally versal. Then the Kodaira-Spencer map*

$$\mathrm{Der}_{(\Sigma,0)}(\mathcal{O}_{\bar{S}},\mathbb{C}) \longrightarrow \mathrm{Ex}_{(\Sigma,0)}(\bar{a}/\bar{S},\mathbb{C}) = \mathrm{Ex}_{(\Sigma,0)}(a_0/S_0,\mathbb{C})$$

is surjective. Conversely, if for $\bar{a} \in \mathbf{F}(\bar{S})$ *the Kodaira-Spencer map above is surjective and* $\bar{S}$ *is smooth over* $\hat{\Sigma}$ *then* $\bar{a}$ *is formally versal.*

Proof. This follows from the Kodaira-Spencer sequence

$$\cdots \to \mathrm{Der}_{(\Sigma,0)}(\mathcal{O}_{\bar{S}},\mathbb{C}) \to \mathrm{Ex}_{(\Sigma,0)}(\bar{a}/\bar{S},\mathbb{C}) \to \mathrm{Ex}_{(\Sigma,0)}(\bar{a},\mathbb{C}) \to \mathrm{Ex}_{(\Sigma,0)}(\bar{S},\mathbb{C}),$$

the criterion 3.4.15 and the fact that $\mathrm{Ex}_{(\Sigma,0)}(\bar{S}/,\mathbb{C})$ vanishes if $\bar{S}$ is smooth over $\hat{\Sigma}$. □

We give a simple application. An object $a_0 \in \mathbf{F}(s_0)$ is called *rigid* resp. *formally rigid*, if the trivial deformation $a_0 \hookrightarrow a_0$ over s_0 is versal resp. formally versal. With other words, every formal deformation of a_0 is isomorphic to the trivial one.

Corollary 3.4.18. a_0 *is formally rigid iff* $\mathrm{Ex}_{(\Sigma,0)}(a_0/S_0,\mathbb{C}) = 0$.

Proof. This follows from the exact sequence

$$0 = \mathrm{Der}_{(\Sigma,0)}(\mathbb{C},\mathbb{C}) \to \mathrm{Ex}_{(\Sigma,0)}(a_0/S_0,\mathbb{C}) \to \mathrm{Ex}(a_0,\mathbb{C}) \to \mathrm{Ex}(\{0\}/\{0\},\mathbb{C}) = 0$$

and the criterion above, where $\{0\}$ stands for the simple point. □

Applying this to deformations of complex spaces, see 3.1.6 (1), this gives the following criterion.

COROLLARY 3.4.19. *If* $X \to \Sigma$ *is smooth and* $H^1(X, \Theta_{X/\Sigma}) = 0$ *then* X *is formally rigid.*

PROOF. By 3.3.7 (1) the infinitesimal deformations $\mathcal{X} \to \mathrm{Spec}(\mathbb{C}[\varepsilon])$ (as a space over Σ) of X are just the extensions $\mathrm{Ex}_\Sigma(X, \mathcal{O}_X)$, and

$$\mathrm{Ex}_\Sigma(X, \mathcal{O}_X) \cong H^1(X, \Theta_{X/\Sigma})$$

by 3.3.10. □

EXAMPLE 3.4.20. We emphasize that in general a formally versal deformation is not versal. As a simple example consider the case that $\Sigma = 0$ is a simple point and $X = \mathbb{C}$ which is a Stein manifold and so is even formally rigid by the preceding corollary, but which is not rigid in the sense above. E.g. the family

$$\mathcal{X} = \{(z,t) \in \mathbb{C} \times \mathbb{C} : |zt| < 1\}$$

with respect to the second projection onto $\mathbb{C}$ cannot be induced from the trivial family $X \to 0$, as the fibers of $\mathcal{X} \to \mathbb{C}$ over points $t \in \mathbb{C}^*$ are discs and so are not biholomorphic to $\mathbb{C}$ by Liouville's theorem. In particular, $X \to 0$ is formally versal but not versal.

3.5. The Theorem of Schlessinger

In the following we fix a local deformation theory $p : \mathbf{F} \to \ \mathbf{An}_{(\Sigma,0)}$. As usual our germs will alwys have base point 0 which we consider at the same time as the germ consisting of a simple point. The central result of this section due to Schlessinger is that every object $a_0 \in \mathbf{F}(0)$ admits a formal versal deformation under some mild hypothesis. Moreover we will compare different (formally) versal deformations. We will show that two of them always differ by a smooth factor.

As already done before, given $a \in \mathbf{F}$ and an extension $(a \hookrightarrow b, u)$ of a by some coherent module we briefly write b for that extension if the other data are clear from the context.

LEMMA 3.5.1. *Let* a *be an object in* $\mathbf{F}(S)$ *defined over some germ* $S = (S, 0)$*. Assume that* $\mathrm{Ex}_{(\Sigma,0)}(a, \mathbb{C})$ *is finite dimensional over* $\mathbb{C}$ *and set* $V := \mathrm{Ex}_{(\Sigma,0)}(a, \mathbb{C})^\vee$*. Then for every finite dimensional vector space* W *over* $\mathbb{C}$ *there is a functorial isomorphism*

$$\mathrm{Ex}_{(\Sigma,0)}(a, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V, W).$$

If $a' \in \mathrm{Ex}_{(\Sigma,0)}(a, V)$ *is the extension corresponding to* $id_V \in \mathrm{Hom}_{\mathbb{C}}(V, V)$ *then the following hold.*

(1) Universal property. *For* $[b] \in \mathrm{Ex}_{(\Sigma,0)}(a, W)$ *there exists a unique linear map* $f_b : V \to W$ *such that* $f_{b^*}(a') \cong b$.

(2) *Let* $j : a' \hookrightarrow a''$ *be a morphism of extensions of* a*. Then there is a retraction* $\gamma : a'' \to a'$*, i.e.* $\gamma j = id_{a'}$.

PROOF. By 3.3.3 (2)

$$\mathrm{Ex}_{(\Sigma,0)}(a, W) \cong \mathrm{Ex}_{(\Sigma,0)}(a, \mathbb{C}) \otimes W = V^\vee \otimes W \cong \mathrm{Hom}(V, W),$$

and on the level of extensions the isomorphism is as indicated. This proves (1).

To show (2) observe first that by the universal property of a' there is a morphism $\gamma_1 : a'' \to a'$ of extensions of a. By the uniquenes statement in (1) the composition

$\gamma_1 j : a' \to a'' \to a'$ is an isomorphism. Now $\gamma := (\gamma_1 j)^{-1} \circ \gamma_1$ is the desired retraction. $\square$

In the following we will call a' the *universal extension* of a and denote it by $\mathrm{ex}(a)$.

REMARKS 3.5.2. (1) In particular, under the assumptions of the lemma, there is always a morphism $b \to a'$ in $\mathbf{Ex}_{(\Sigma,0)}(a)$. Observe that this morphism is only determined up to an automorphism of b in $\mathbf{Ex}_{(\Sigma,0)}(a)$.

(2) In the situation of the lemma, the induced map

$$\mathrm{Ex}_{(\Sigma,0)}(a, W) \longrightarrow \mathrm{Ex}_{(\Sigma,0)}(a', W)$$

is zero as follows from (2).

(3) It is clear from the proof that the above lemma also holds for formal objects $\bar{a} \in \hat{\mathbf{F}}(\bar{S})$ defined over some formal germ $\bar{S}$.

In the following, a deformation a of $a_0 \in \mathbf{F}(0)$ will be called *versal up to order* n if the condition (FV) in 3.4.6 is satisfied in the case that $T \hookrightarrow T' := p(b')$ is an extension of fat point with $\mathfrak{m}_{T'}^{n+1} = 0$.

Let $a_0 \in \mathbf{F}(0)$ be given and assume that $\mathrm{Ex}_{(\Sigma,0)}(a_0, \mathbb{C})$ is finite dimensional. Let $\bar{a} = (a_n)$ be the formal deformation defined inductivily by $a_{n+1} := \mathrm{ex}(a_n)$, where $a_n \to \mathrm{ex}(a_n)$ is *a* universal extension of a_n. That this construction is well defined is seen by the following lemma.

LEMMA 3.5.3. (1) *For every n, the vector space $\mathrm{Ex}_{(\Sigma,0)}(a_n, \mathbb{C})$ has finite dimension.*

$(2)_n$ *a_n is versal up to order n.*

$(3)_n$ *If S_n is the fat point underlying a_n then $\mathcal{O}_{S_{n-1}} = \mathcal{O}_{S_n}/\mathfrak{m}_{S_n}^n$, where $\mathfrak{m}_{S_n} \subseteq \mathcal{O}_{S_n}$ denotes the maximal ideal.*

(4) *If $\bar{S} = \varinjlim S_n$ is the associated formal complex space then the complete local $\mathbb{C}$-algebra $\mathcal{O}_{\bar{S}}$ is isomorphic to a quotient of $\mathcal{O}_{\hat{\Sigma}}[\![V]\!]$, with $V := \mathrm{Ex}_{(\Sigma,0)}(a_0, \mathbb{C})^\vee$, modulo an ideal $I \subseteq \mathfrak{m}^2_{\mathcal{O}_{\hat{\Sigma}}[\![V]\!]} + \mathfrak{m}_{\hat{\Sigma}}[\![V]\!]$.*

PROOF. The proof of (1) follows from the Kodaira-Spencer sequence

$$\cdots \to \mathrm{Ex}_{(\Sigma,0)}(a_n/S_n, \mathbb{C}) \to \mathrm{Ex}_{(\Sigma,0)}(a_n, \mathbb{C}) \to \mathrm{Ex}_{(\Sigma,0)}(S_n, \mathbb{C})$$

and the fact that

$$\mathrm{Ex}_{(\Sigma,0)}(a_n/S_n, \mathbb{C}) \cong \mathrm{Ex}_{(\Sigma,0)}(a_0/S_0\mathbb{C}) \cong \mathrm{Ex}_{(\Sigma,0)}(a_0, \mathbb{C}),$$

see 3.4.14. For the proof of $(2)_n$ and $(3)_n$ we proceed by induction on n. For $n = 1$ $(2)_n$ follows from 3.5.1 whereas $(3)_n$ is trivial. Assume that $n > 1$ and that $(2)_{n-1}$ and $(3)_{n-1}$ are satisfied. We show first that $(3)_n$ holds. Consider S'_{n-1} with $\mathcal{O}_{S'_{n-1}} = \mathcal{O}_{S_n}/\mathfrak{m}_{S_n}^n$ which fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] & \mathcal{O}_{S'_{n-1}} \arrow[r] & \mathcal{O}_{S_{n-1}} \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u] & \mathcal{O}_{S_n} \arrow[r] \arrow[u] & \mathcal{O}_{S_{n-1}} \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd}$$

Let a'_{n-1} be the object in $\mathbf{F}(S'_{n-1})$ induced by $S'_{n-1} \to S_n$ from a_n, so that $a_{n-1} \hookrightarrow a'_{n-1}$ is an extension of a_{n-1} by W, i.e. $[a'_{n-1}] \in \mathrm{Ex}_{(\Sigma,0)}(a_{n-1}, W)$ Since a_{n-1} is versal up to order $n-1$ there is a section $a'_{n-1} \to a_{n-1}$ of $a_{n-1} \hookrightarrow a'_{n-1}$, i.e. $[a'_{n-1}] = 0$ in $\mathrm{Ex}_{(\Sigma,0)}(a_{n-1}, W)$. Hence the map

$$V = \mathrm{Ex}_{(\Sigma,0)}(a_{n-1}, \mathbb{C})^\vee \longrightarrow W$$

corresponding to a'_{n-1} is zero by 3.5.1. Since $V \to W$ is surjective, it follows that W is zero and so $S'_{n-1} = S_{n-1}$ as desired.

For the proof of $(2)_n$ we consider a diagram of solid arrows in $\mathbf{F}$

$$\begin{tikzcd}
b \arrow[r] \arrow[d] & a_n \arrow[d, equal] \\
b' \arrow[r, dotted] & a_n
\end{tikzcd}$$

where $b \hookrightarrow b'$ is an extension over an extension of fat points $T \hookrightarrow T'$ with $\mathfrak{m}_{T'}^{n+1} = 0$. We must show that there is a morphism as indicated by the dotted arrow making the diagram commutative. Set $a' := a_n \amalg_b b'$ so that the canonical morphism $j : a_n \hookrightarrow a'$ is an extension over, say $S_n \hookrightarrow S'$. As $\mathcal{O}_{S'}$ is the fibred product of $\mathcal{O}_{S_n}$ and $\mathcal{O}_{T'}$ over $\mathcal{O}_T$ we we have also $\mathfrak{m}_{S'}^{n+1} = 0$. It is sufficient to verify that j admits a retraction $\gamma : a' \to a_n$. Using 3.5.1 (2) we need to show that the composed morphism $a_{n-1} \hookrightarrow a'$ is an extension, i.e. that the kernel, say I, of the underlying map

$$\mathcal{O}_{S'} \xrightarrow{j^*} \mathcal{O}_{S_n} \xrightarrow{can} \mathcal{O}_{S_{n-1}}$$

is of square zero. Take $x \in I$. Using $(3)_{n-1}$ we get that $j^*(x) \in \mathfrak{m}_{S_n}^n$. By the surjectivity of j^* we find an element $y \in \mathfrak{m}_{S'}^n$ mapping to $j^*(x)$, i.e. $x - y \in \operatorname{Ker} j^*$. As this is an ideal of square 0 we get $0 = (x-y)^2 = x^2 + 2xy + y^2$. But $2xy$ and y^2 are contained in $\mathfrak{m}_{S'}^{n+1} = 0$ and so $x^2 = 0$. Hence $I^2 = 0$ and $(2)_n$ follows.

By construction, $\mathcal{O}_{S_1} \cong \mathbb{C}[\![V]\!]/(V^2)$ and by (3) $\mathcal{O}_{S_n}/\mathfrak{m}_{S_n}^2 \cong \mathcal{O}_{S_1}$. Hence (4) follows. □

THEOREM 3.5.4 (Schlessinger). *Let $a_0 \in \mathbf{F}(0)$ and assume that the vectorspace $V := \mathrm{Ex}_{(\Sigma,0)}(a_0, \mathbb{C})$ is of finite dimension. Then a_0 admits a formal deformation $\bar{a}$ which is formally versal.*

PROOF. Let $\bar{a} = (a_n)$ be as in 3.5.3. Because of loc.cit. (4) $\bar{a}$ is an object in $\hat{\mathbf{F}}$, and because of (2) it satisfies the property (FV) in 3.4.3. □

As the construction shows the basis $\bar{S}$ of the formally versal deformation $\bar{a}$ constructed in 3.5.4, has tangent space $\mathrm{Ex}\,(\Sigma, 0)(a_0, \mathbb{C})$. This deformation has the following minimality property.

PROPOSITION 3.5.5. *Every formal deformation $\bar{b}$ of a_0 is induced from $\bar{a}$ by a map $\bar{T} := p(\bar{b}) \xrightarrow{f} \bar{S}$, i.e. $f^*(\bar{a}) \cong \bar{b}$. Moreover, the associated map of tangent spaces*

$$T_0(\bar{T}) \longrightarrow T_0(\bar{S})$$

is uniquely determined by $\bar{b}$.

PROOF. The existence of an f with $f^*(\bar{a}) \cong \bar{b}$ follows from the formal versality of $\bar{a}$. If $\mathcal{O}_{\bar{T}_1} = \mathbb{C}[W]$ then $W^\vee = T_0(\bar{T})$ and similarly $\mathcal{O}_{\bar{S}_1} = \mathbb{C}[V]$ with $V^\vee = T(\bar{S}) = \mathrm{Ex}_{(\Sigma,0)}(a_0, \mathbb{C})$. A map f induces a map $f_1 : \bar{T}_1 \to \bar{S}_1$ which is uniquely determined by the associated map of tangent spaces $df : W^\vee \to V^\vee$. Thus

$$\mathrm{Hom}(\bar{T}_1, \bar{S}_1) = \mathrm{Hom}_{\mathbb{C}}(V, W),$$

and under the identification

$$\mathrm{Ex}_{(\Sigma,0)}(a_0, W) \cong \mathrm{Hom}_{\mathbb{C}}(V, W)$$

the dual of df corresponds uniquely to b_1 over T_1, by 3.5.1. □

DEFINITION 3.5.6. A formal deformation $\bar{a} \in \mathbf{F}(\bar{S})$ of a_0 is *formally semiuniversal* if it is formally versal and if the tangent space of $\bar{S}$ is isomorphic to $\mathrm{Ex}_{(\Sigma,0)}(a_0, \mathbb{C})$. For a (global) deformation theory $p : \mathbf{F} \to \mathbf{An}_\Sigma\mathrm{A}$ a convergent deformation $a \in \mathbf{F}(S)$ of $a_0 \in F(\{s_0\})$ is called *semiuniversal* if a is versal and if $T_{s_0}(S)$ is just $\mathrm{Ex}_{(\Sigma,0)}(a_0, \mathbb{C})$.

PROPOSITION 3.5.7. (1) *A formally semiuniversal deformation of* a_0 *is uniquely determined up to (noncanical) isomorphism.*

(2) *Let* $\bar{a} \in \mathbf{F}(\bar{S})$ *be formally semiuniversal and* $\bar{b} \in \mathbf{F}(\bar{T})$ *a formally versal deformation of* a_0. *Let* $f : \bar{T} \to \bar{S}$ *be a morphism with* $f^*(\bar{a}) \cong \bar{b}$. *Then* f *is smooth, i.e.* $\mathcal{O}_{\bar{T}} = \mathcal{O}_{\bar{S}}[\![T_1, \dots, T_n]\!]$.

PROOF. Observe that (2) implies (1). In order to show (2) we first remark that $Tf : T_0(\bar{T}) \to T_0(\bar{S})$ is surjective. In fact, by the formal versality of $\bar{b}$ there is a morphism $g : \bar{S} \to \bar{T}$ with $g^*(\bar{b}) \cong \bar{a}$, and the composed map $T(f) \circ T(g)$ is the identity on $T_0(\bar{S})$ since $\bar{a}$ is assumed to be formally semiuniversal. Let $\bar{T} \hookrightarrow \bar{T}'$ be an $\bar{S}$ embedding into a space $\bar{T}'$ which is smooth over $\bar{S}$. Since $T_0(\bar{T}) \to T_0(\bar{S})$ is surjective we may assume that $T_0(\bar{T}) \xrightarrow{\sim} T_0(\bar{T}')$.

The morphism $\bar{b} \to \bar{a}$ induces a morphism $j : \bar{b} \hookrightarrow \bar{a} \amalg_{\bar{S}} \bar{T}'$. As $\bar{b}$ is semiuniversal there is a section $\sigma : \bar{a} \amalg_{\bar{S}} \bar{T}' \to \bar{b}$, i.e. $\sigma j = id_{\bar{b}}$. Hence $\bar{T}$ is a retract of $\bar{T}'$ and so $\bar{T}$ and $\bar{T}'$ must be isomorphic having isomorphic tangent spaces. □

PROPOSITION 3.5.8. *Assume that there exists a versal deformation of* a_0 *in* $\mathbf{F}$. *Then the following hold.*

(1) *There exists a semiuniversal deformation of* a_0 *in* $\mathbf{F}$.
(2) *Every deformation of* a_0 *in* $\mathbf{F}$ *which formally versal is also versal.*

PROOF. First we show (1). Let $a \in \mathbf{F}(S, 0)$ be a versal deformation of a_0 and $\bar{b} \in \hat{\mathbf{F}}(\bar{T})$ a formally semiuniversal deformation of a_0. Then the completion $\hat{a}$ is induced by a map $f : \hat{S} \to \bar{T}$, and by 3.5.6 f is smooth, i.e.

$$\mathcal{O}_{\hat{S}} = \mathcal{O}_{\bar{T}}[\![X_1, \dots, X_n]\!].$$

We may assume that $X_1, \dots X_n$ are already in $\mathcal{O}_{S,0}$. Let $T \subseteq S$ be the subspace given by $X_1 = \cdots = X_n = 0$, and let $b \in \mathbf{F}(T, 0)$ be the object induced from $a \in \mathbf{F}(S, 0)$. By construction $\hat{b} \cong \bar{b}$ and $\hat{T} \cong \bar{T}$. We now show that $(S, 0) \cong (T \times \mathbb{C}^n, 0) =: (S', 0)$ and $a' := b \times \mathbb{C}^n \cong a$. Namely, let S_1' and S_1 be the first infinitesimal neighbourhoods. Then the induced infinitesimal deformations $a_1' \in \mathbf{F}(S_1')$ and $a_1 \in \mathbf{F}(S_1)$ are isomorphic under an isomorphism, say $f : a_1' \to a_1$. In the diagram

$$\begin{tikzcd} a_1' \arrow[r, "f"] \arrow[d] & a_1 \arrow[d] \\ a' \arrow[r, dotted, "g"] & a \end{tikzcd}$$

there is a lifting g , and the map $T(p(g)) : T_0(S') \to T_0(S)$ is an isomorphism by construction and so $p(g) : S' \to S$ is an embedding.

On the other hand, $\hat{S}'$ and $\hat{S}$ are (abstractly) isomorphic since the local rings $\mathcal{O}_{\hat{S}'}, \mathcal{O}_{\hat{S}}$ are both formal power series rings in n variables over $\mathcal{O}_{\bar{T}}$. This implies (e.g. by observing that the surjective maps $\mathcal{O}_{S_n} \to \mathcal{O}_{S'_n}$ induced by $p(g)$ are bijective by length reasons) that $p(g) : S' \to S$ is an isomorphism. Thus 3.5.8 follows from 3.4.10 (2).

In order to show (2) consider a deformation $a \in \mathbf{F}(S,0)$ of a_0 which is formally versal. Let $b \in \mathbf{F}(T,0)$ be a semiuniversal deformation of a_0. Then there is a morphism $f : (S,0) \to (T,0)$ with $f^*(b) \cong a$. By 3.5.6 the map $\hat{S} \to \hat{T}$ is smooth and so f is smooth too. Applying again 3.4.10 (2) we obtain that a is versal. $\square$

CHAPTER 4

Applications to Unfoldings

In the nowadays classical theory of singularities of (holomorphic) mappings $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ one considers various equivalence relations given by groups of automorphisms which act on such germs. For instance, in sect. 1.2 we already treated contact equivalence and gave explicit criteria when two germs are equivalent under this relation. In this chapter we present a systematic account of this theory from the point of view of deformations and give a unified treatment of the so called standard theorems, see e.g. [**AVGL**]. The theory of unfoldings developed independently from deformation theory in the late sixties. It became apparent soon afterwards that unfoldings can be viewed as (unobstructed) deformations, see e.g. [**Tei**].

In our approach which follows closely [**BFl**] the key tool underlying all proofs is the *relative Kodaira-Spencer map* of a deformation which was introduced in sect. 2.3. Its vanishing for a 1-parameter family signifies that the deformation is trivial whereas the surjectivity for a deformation over a smooth base characterizes versality. Of course the Kodaira-Spencer map appears in various disguises already in the classical proofs, e.g. as "homological equation" in [**AVGL**, Chapt.3, 1.5]. Systematic use of the Kodaira-Spencer class clarifies the proofs considerably and allows for generalizations. As a typical example we give in sect. 3.3 a simplified and more conceptual proof of the theorems of Mather-Yau type [**MYa**], [**GHa**]. We deduce these results for mapping germs $(X, 0) \to (\mathbb{C}^p, 0)$ where $(X, 0)$ is an arbitrary germ of a complex space.

4.1. Unfoldings and Deformations

Definition 4.1.1. Let $(X, 0)$, $(Y, 0)$ be germs of complex spaces and $f : (X, 0) \to (Y, 0)$ be a holomorphic map. An *unfolding* of f over the base $(S, 0)$ is a morphism

$$F : (X \times S, 0) \longrightarrow (Y, 0)$$

such that $F(x, 0) = f(x)$ for x in a neighbourhood of $0 \in X$.

Sometimes in the literature the corresponding S-morphism

$$(F, id_S) : (X \times S, 0) \longrightarrow (Y \times S, 0)$$

is called an unfolding. An unfolding F must be considered as a deformation over the parameter space S. The object which is deformed is in this case not just a space but a mapping, namely f. This viewpoint will be made more precise below.

Example 4.1.2. As a simple example consider the function $f : \mathbb{C} \to \mathbb{C}$ with $f(x) = x^n$, i.e. $(X, 0) = (Y, 0) = (\mathbb{C}, 0)$. An unfolding of f is given by the "universal" polynomial

$$F(x, a) := x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

where $a = (a_0, \dots, a_{n-1}) \in S := \mathbb{C}^n$. Later on we will see that every other unfolding of f can be induced from F if we allow certain S-automorphisms.

In general there are several natural equivalence relations on unfoldings which act by automorphisms. In sect. 1.2 we studied already contact equivalence in the absolute case, i.e. $S = 0$. The same relation will also be investigated for unfoldings. Another possibility is to let S-automorphisms of $(X \times S, 0)$ act on the right of $F(x, s)$ or S-automorphisms of $(Y \times S, 0)$ act on the left of $(F \times id_S)$, or to combine these two actions. Even more generally, we will admit quite general actions of groups on the mapping germs, which is the proper framework for such equivalence relations. For this we introduce the following notations.

4.1.3. Let in the following $(X, 0)$ and $(Y, 0)$ be fixed and set

$$E(S, 0) := \mathrm{Mor}((X, \times S, 0), (Y, 0))$$

be the set of morphisms $F : (X \times S, 0) \to (Y, 0)$. Obviously $(S, 0) \mapsto E(S, 0)$ gives a set valued functor $E : \mathbf{Germs} \to \mathbf{Sets}$. The elements of $E(S, 0)$ are also called *unfoldings* over $(S, 0)$.

It is useful to note that the functor E is compatible with fibred sums: let $(S, 0) \hookrightarrow (S', 0)$ be an extension and $(T, 0) \to (S, 0)$ a finite morphism of germs. Denoting by $T' := S' \amalg_S T$ the fibred sum we get

$$(*) \qquad E(T', 0) = E(S', 0) \times_{E(S,0)} E(T, 0).$$

Indeed, if $F' \in E(S', 0)$ and $H \in E(T, 0)$ are unfoldings inducing the same unfolding, say F, on $(S, 0)$ then we can form the fibred sum $F' \amalg_F H$. Using the fact that fibred sums are compatible with taking products it is seen that $F' \amalg_F H$ represents an element in $E(T', 0)$ inducing F' on $(S', 0)$ and H on $(T, 0)$.

4.1.4. Assume that we are given a group-valued functor

$$G : \mathbf{Germs}^{op} \longrightarrow \mathbf{Groups}$$

$$(S, 0) \longmapsto G(S, 0),$$

together with an action of G on E. This means that for every germ $(S, 0)$ the group $G(S, 0)$ acts on $E(S, 0)$, and for every morphism $(S', 0) \to (S, 0)$ the induced diagram

$$\begin{tikzcd}
G(S,0) \times E(S,0) \arrow[r] \arrow[d] & E(S,0) \arrow[d] \\
G(S',0) \times E(S',0) \arrow[r] & E(S',0)
\end{tikzcd}$$

commutes.

Let $\mathbf{E}_G(S, 0)$ be the groupoid associated to the $G(S, 0)$-set $E(S, 0)$, see 3.1.7. More explicitly, the objects are the elements of $E(S, 0)$, and for $F_1, F_2 \in E(S, 0)$ the morphisms from F_1 to F_2 are all elements g of $G(S, 0)$ with $g.F_1 = F_2$. Two germs F_1, F_2 are called *G-equivalent* if F_1 and F_2 are in the same orbit under $G(S, 0)$. The reader may easily verify that this defines a fibration in groupoids $\mathbf{E}_G \to \mathbf{Germs}$ with fibers $\mathbf{E}_G(S, 0)$.

In singularity theory the following examples are of importance.

Examples 4.1.5. (1) (Right-equivalence). For a germ $S = (S, 0)$ let $\mathcal{R}_e(S, 0)$ be the group $\mathrm{Aut}_{(S,0)}(X \times S, 0)$ of all $(S, 0)$ automorphisms of $(X \times S, 0)$. This group acts naturally on $E(S, 0)$. More concretely, if F is an unfolding and $g \in \mathcal{R}_e(S, 0)$ then $g.F := F \circ g^{-1}$. This defines a fibration in groupoids $\mathbf{E}_{\mathcal{R}_e} \to \mathbf{Germs}$. We define $\mathcal{R}(S, 0)$ to be the subgroup of all those g in $\mathcal{R}(S, 0)$ with $g|(0 \times S) = id_{0\times S}$. The associated equivalence relations are called *extended right equivalence* resp. *right equivalence*.

(2) (Left-equivalence). For a germ $(S, 0)$ the group of automorphisms $\mathcal{L}_e(S, 0) := \mathrm{Aut}_{(S,0)}(Y \times S, 0)$ acts naturally on $E(S, 0)$ from the left by

$$(g.F, id_S) := g \circ (F, id_S).$$

In this way one obtains a fibration in groupoids $\mathbf{E}_{\mathcal{L}} \to \mathbf{Germs}$. The associated equivalence relation is called *extended left equivalence*. Restricting to elements g of $\mathcal{L}_e(S, 0)$ with $g|(Y \times S) = id_{Y\times S}$ defines again a subgroup $\mathcal{L}(S, 0)$ and leads to the *left equivalence* of germs.

(3) (Right-left-equivalence). Combining the actions of $\mathcal{R}, \mathcal{L}$ on E in (1), (2) we obtain an action of $\mathcal{A} = \mathcal{R} \times \mathcal{L}$ on E and so a fibration in groupoids $\mathbf{E}_{\mathcal{A}}$. The associated equivalence relation is called *right-left-equivalence* or $\mathcal{A}$-equivalence. Similarly we can form $\mathcal{A}_e = \mathcal{R}_e \times \mathcal{L}_e$ and obtain the "extended" version of right-left equivalence.

(4) (Contact equivalence). In this example we restrict ourselves to the case that $(Y, 0) = (\mathbb{C}^p, 0)$ is smooth. For a germ $(S, 0)$ let $\mathcal{K}_e(S, 0)$ be the group of contact-equivalences which is defined to be the semidirect product

$$\mathcal{K}_e(S, 0) := \mathcal{R}_e(S, 0) \ltimes GL_p(\mathcal{O}_{X\times S,0}),$$

where the group structure is given by

$$(\varphi, M)(\psi, N) := \big(\varphi \circ \psi, M(N \circ \varphi^{-1})\big)$$

for $\varphi, \psi \in \mathcal{R}_e(S, 0)$ and $M, N \in GL_p(\mathcal{O}_{\mathbb{C}^n\times S,0})$. This group acts on $E(S, 0)$ by

$$(\varphi, M)F := M(F \circ \varphi^{-1}).$$

This gives the so called $\mathcal{K}_e$-equivalence or *extended contact-equivalence*. It is an easy exercise left to the reader to verify that two unfolding F_1, F_2 are $\mathcal{K}_e$-equivalent iff the analytic germs $F_1^{-1}(0)$ and $F_2^{-1}(0)$ are $(S, 0)$-isomorphic.

Similarly, taking the subgroup $\mathcal{K}(S, 0) := \mathcal{R}(S, 0) \ltimes GL_p(\mathcal{O}_{X\times S,0})$ of $\mathcal{K}_e$ leads to $\mathcal{K}$- or contact equivalence. Moreover, two unfolding F_1, F_2 with $0 \times S \subseteq F_1^{-1}(0)$, $F_2^{-1}(0)$ are $\mathcal{K}$-equivalent iff there is an S-automorphism of germs $F_1^{-1}(0) \to F_2^{-1}(0)$ inducing the identity on $S \times 0$.

(5) ($\mathcal{C}$-equivalence) We again assume that $(Y, 0) = (\mathbb{C}^p, 0)$ is smooth. Restricting the action in (4) to the subgroup $GL_p(\mathcal{O}_{X\times S,0})$ of $\mathcal{K}_e(S, 0)$ we get the so called $\mathcal{C}$-equivalence. Two mapping germs $F = (F_1, \ldots, F_p)$ and $G = (G_1, \ldots, G_p)$ in $E(S, 0)$ are obviously $\mathcal{C}$-equivalent iff the ideals $(F_1, \ldots, F_p)$ and $(G_1, \ldots, G_p)$ in $\mathcal{O}_{X\times S,0}$ are equal.

Note that in the absolute case, i.e. $S = 0$ is a simple point, the extended and non extended equivalence relations coincide. In many of the applications we will only deal with the absolute case. But even here, in the proofs there occur the corresponding relations for unfoldings, and one has to distingush carefully the extended and non extended versions!

We also remark that one can extend the notion of $\mathcal{K}$- and $\mathcal{C}$-equivalence to the case that $(Y,0)$ is a singular germ, see [**BFl**, 1.3 (4) and (5)] for details.

Examples 4.1.6. (1) It is helpful to consider the simplest case, namely that of a polynomial $f(z) \neq 0$ of one variable z where $S = 0$. We can find a unit $\varepsilon \in \mathcal{O}_{\mathbb{C},0}$ such that $\varepsilon(z)f(z) = z^n$ for some $n \in \mathbb{N}$, With other word, $f(z)$ is $\mathcal{C}$-equivalent and then also $\mathcal{K}$-equivalent to z^n. With the new coordinate $z' := (\varepsilon)^{-1/n}z$ we can write $f(z) = z'^n$. Therefore f and z^n also are right equivalent. On the other hand, there are many polynomials which are not left equivalent to a monomial, take e.g. $z^2 + z^3$.

(2) Let us treat the case of an unfolding $F : (\mathbb{C} \times S \to \mathbb{C})$ of a function $F(z,0) \neq 0$ of one variable. By the preparation theorem of Weierstra there is a unit $u \in \mathcal{O}_{\mathbb{C}\times S,0}$ and a Weierstra polynomial

$$P = z^n + a_{n-1}(s)z^{n-1} + \cdots + a_0(s)$$

such that $F = uP$. In particular F and P are $\mathcal{C}$-equivalent and so in particular $\mathcal{K}$-equivalent. Later on we will see that F is already right-equivalent to such a polynomial, see 4.5.8.

We will now examine the homogeneity of these fibrations in groupoids. We again consider the general situation of 4.1.4

4.1.7. Assume that the action of G on E satisfies the following properties.

(1) If $(S,0) \hookrightarrow (S',0)$ is an extension then $G(S',0) \to G(S,0)$ is surjective.
(2) Let

$$\begin{tikzcd}
(S,0) \arrow[r, "i"] \arrow[d, "\alpha"'] & (S',0) \arrow[d, "\beta"] \\
(T,0) \arrow[r, "j"] & (T',0)
\end{tikzcd}$$

be a cocartesian diagram of germs of complex spaces such that $(S,0) \to (T,0)$ is finite and $(S,0) \hookrightarrow (S',0)$ is an extension. Then $G(T',0)$ is the fibered product of $G(S'0)$ and $G(T,0)$ over $G(S,0)$.

We will show

Proposition 4.1.8. *If 4.1.7 is satisfied then* $\mathbf{E}_G \to$ **Germs** *is a deformation theory.*

Proof. In the situation of 4.1.7 (2) let $F' \in E(S',0)$, $H \in E(T,0)$ with $F := \alpha^*(H) = g.i^*(F')$ for some $g \in G(S,0)$. We have to construct the fibered sum of F' and H over F in $\mathbf{E}_G$. By property (1) we can find $g' \in G(S',0)$ lifting g. Replacing F' by $g'.F'$ we may assume that $F = \alpha^*(H) = i^*(F')$. Using the fact that the functor E is compatible with fibred sums, see 4.1.3 $(*)$, we can form the fibred sum $H' := F' \amalg_F H$ which is a map

$$H' : (X \times T',0) \to (Y \times T',0),$$

i.e. $H' \in E(T',0)$. We claim that H' represents the fibred sum of F' and H in $\mathbf{E}_G$. For this consider in $\mathbf{E}_G$ a diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F' \arrow[d] & \text{over} & S \arrow[r, "i"] \arrow[d, "\alpha"'] & S' \arrow[d, "\beta"] \\ H \arrow[r] & L & & T \arrow[r, "j"] & T'. \end{tikzcd}$$

By the lemma below, it is sufficient to show that there is a morphism $H' \to L$ over $id_{T'}$ in $\mathbf{E}_G$ which induces on F' and H the given arrows.

The morphisms $H \to L$ and $F' \to L$ are given by elements $h \in G(T,0)$ and $f' \in G(S',0)$ with $h.H = j^*(L)$ and $f'.F' = \beta^*(L)$. It follows that

$$i^*(f').F = i^*(f'.F') = i^*\beta^*(L) = \alpha^*j^*(L) = \alpha^*(h.H) = \alpha^*(h).F.$$

Hence (f', h) defines an element in $G(S',0) \times_{G(S,0)} G(T,0)$. Because of our assumption (2) there is a unique element $h' \in G(T',0)$ with $j^*(h') = h$ and $\beta^*(h') = f'$. Obviously $h'.H' = L$. $\square$

In the above proof we have used the following simple observation.

Lemma 4.1.9. *Let $p : \mathbf{F} \to \mathbf{C}$ be a cofibration in groupoids and consider a diagram in* $\mathbf{F}$

$$\begin{tikzcd} a \arrow[r] \arrow[d] & a' \arrow[d] & \textit{over} & S \arrow[r, "i"] \arrow[d, "\alpha"'] & S' \arrow[d, "\beta"] \\ b \arrow[r] & b' & & T \arrow[r, "j"] & T'. \end{tikzcd}$$

such that T' is the fibred sum $S' \amalg_S T$. Then b' represents the fibred sum of a', b over a iff the following condition is satisfied.

(*) For every pair of morphisms $\beta_\sharp : a' \to c$ over β and $j_\sharp : b \to c$ over j in $\mathbf{F}$ inducing the same morphism on a there is a unique morphism $b' \to c$ over $id_{T'}$ such that the compositions $a' \to b' \to c$ and $b \to b' \to c$ are the given arrows $\beta_\sharp$, $j_\sharp$, respectively.

Proof. Clearly $(*)$ is a special case of the universal property for fibred sums. Assume that $(*)$ holds and take morphisms $a' \to d$, $b \to d$ which yield the same morphism on a. Then the arrows $S' = p(a') \to p(d)$ and $T = p(b) \to p(d)$ define a morphism of the fibred sum $\gamma : T' \to p(d)$. Applying our assumption $(*)$ to $c := \gamma^*(d)$ gives a morphism $b' \to c$ over $id_{T'}$. Composing with the natural morphism $c \to d$ we get an arrow $b' \to d$ such that the the compositions $a' \to b' \to d$ and $b \to b' \to d$ are the given morphisms $a' \to d$, $b \to d$ respectively. $\square$

Corollary 4.1.10. *The examples (4.1.5) 1–5 are deformation theories.*

Proof. We will show this for the case of extended right-equivalence, see (4.1.5) (1); the proofs in the other cases are similar. By definition, the group $\mathcal{R}_e(S,0)$ is the set of all automorphisms $(X \times S, 0) \to (X \times S, 0)$. In the situation of 4.1.7 (2), let $g' \in \mathcal{R}_e(S',0)$, $h \in \mathcal{R}_e(T,0)$ be right equivalences inducing the same element, say g, in $\mathcal{R}_e(S,0)$. As fibred sums are compatible with products the map $g' \amalg_g h$ defines an element of $\mathcal{R}_e(T',0)$ pulling back to g', h, respectively. Hence 4.1.7 (2) is satisfied. Finally, that (1) is fulfilled follows from the smoothness principle, see **??**. $\square$

4.2. Infinitesimal Computations

4.2.1. Let $(X,0)$ be a germ of complex spaces and consider the functor E : **Germs** $\to$ **Sets** which associates to $(S,0)$ the set of all S-morphisms $F:(X\times S,0)\to(\mathbb{C}^p,0)$, i.e. we consider in 4.1.3 the situation that $(Y,0)=(\mathbb{C}^p,0)$ is a smooth germ. Such a map is uniquely determined by its components. With other words, the elements of $E(S,0)$ are in a 1-1 correspondence with the elements of $\mathfrak{m}^p_{X\times S,0}$ where $\mathfrak{m}_{X\times S,0}$ denotes the maximal ideal of $\mathcal{O}_{X\times S,0}$. Assume that there is a homogeneous group valued functor G acting on E and satisfying 4.1.7 such that we get a deformation theory $\mathbf{E}_G\to$ **Germs**.

4.2.2. Let $F:(X\times S,0)\to(\mathbb{C}^p,0)$ be an unfolding and $\mathcal{M}$ a coherent $\mathcal{O}_S$-module. In the following it is convenient to write the structure sheaf of the trivial extension $S[\mathcal{M}]$ as $\mathcal{O}_{S[\mathcal{M}]}=\mathcal{O}_S\oplus\varepsilon\mathcal{M}$ where $\varepsilon^2=0$. We set $\mathcal{M}_X:=\mathcal{O}^p_{X\times S,0}\otimes\mathcal{M}$ so that $X\times S[\mathcal{M}]\cong(X\times S)[\mathcal{M}_X]$. The unfoldings

$$\tilde{F}\in E(S[\mathcal{M}],0)\cong\mathfrak{m}^p_{X\times S[\mathcal{M}],0}=\mathfrak{m}^p_{X\times S,0}\oplus\varepsilon\mathcal{M}^p_X$$

restricting to F in $\mathcal{O}^p_{X\times S,0}$ are those of the form $\tilde{F}=F-\varepsilon H$ with $H\in\mathcal{M}^p_X$. This simple observation will allow us to compute very explicitly the extension functors (see 3.3.1)

$$\mathrm{Ex}_G(F/S,\mathcal{M}):=\mathrm{Ex}_{\mathbf{E}_G}(F/S,\mathcal{M})$$

for the deformation theory of unfoldings modulo G-equivalence. First note that by the above argument we have a natural surjective map of $\mathcal{O}_{S,0}$-modules

$$\mathcal{M}^p_X\to\mathrm{Ex}_G(F/S,\mathcal{M})\ .$$

Our task is to determine the kernel of this map. An element $\tilde{F}=F-\varepsilon H$ is isomorphic in $\mathbf{E}_G$ to $F[\mathcal{M}]=F+\varepsilon 0$ iff $\tilde{F}=g.F$ for some $g\in G\left(S[\mathcal{M}],0\right)$ that restricts to the identity in $G(S,0)$. Setting

$$G(\mathcal{M}):=\mathrm{Ker}\left(G(S[\mathcal{M}],0)\to G(S,0)\right)$$

we obtain a map $\gamma=\gamma^F_G:G(\mathcal{M})\to\mathcal{M}^p_X$ via $g\mapsto H$ if $g.F=F-\varepsilon H$. Let us compute the kernel of γ_F. An element g of $G\mathcal{M})$ is in the kernel of γ iff $g.F=F$ which just means that g gives an automorphism of F in $\mathbf{E}_G$ inducing the identity on $S[\mathcal{M}]$. Therefore the second row in the diagram in 4.2.3 below is exact. Moreover we get:

Proposition 4.2.3. *There is a natural diagram with exact rows*

$$\begin{tikzcd}
\mathrm{Aut}_{\mathbf{E}_G}(F/S,\mathcal{M}) \arrow[r, hook] \arrow[d, equal] & \mathrm{Aut}_{\mathbf{E}_G}(F,\mathcal{M}) \arrow[r] \arrow[d, "\alpha"'] & \mathrm{Der}(\mathcal{O}_{S,0},\mathcal{M}) \arrow[r, "\delta_{KS}"] \arrow[d, "\beta"'] & \mathrm{Ex}_G(F/S,\mathcal{M}) \arrow[d, equal] & \\
\mathrm{Aut}_{\mathbf{E}_G}(F/S,\mathcal{M}) \arrow[r, hook] & G(\mathcal{M}) \arrow[r, "\gamma"] & \mathcal{M}^p_X \arrow[r] & \mathrm{Ex}_G(F/S,\mathcal{M}) \arrow[r] & 0
\end{tikzcd}$$

where the first row is the Kodaira-Spencer sequence. The map β is given by

$$\vartheta\longmapsto\vartheta_X(F)=(\vartheta_X(F_1),\ldots,\vartheta_X(F_p))\ ,$$

with $\vartheta_X\in\mathrm{Der}(\mathcal{O}_{X\times S,0},\mathcal{M}_X)$ the canonical lifting of ϑ.

PROOF. We first show that δ_{KS} factors through β. In terms of analytic algebras the unfolding $F_\vartheta = \delta_{KS}(\vartheta)$ fits into the diagram

$$\begin{tikzcd}
\mathcal{O}_S \arrow[r] \arrow[d, "id-\varepsilon\vartheta"'] & \mathcal{O}_{\mathbb{C}^p\times S,0} \arrow[r, "\circ(F,id_S)"] \arrow[d, "id-\varepsilon\vartheta_Y"] & \mathcal{O}_{X\times S,0} \arrow[d, "id-\varepsilon\vartheta_X"] \\
\mathcal{O}_S[\mathcal{M}] \arrow[r] & \mathcal{O}_{\mathbb{C}^p\times S[\mathcal{M}],0} \arrow[r, "\circ(F_\vartheta,id_S)"] & \mathcal{O}_{X\times S[\mathcal{M}],0}
\end{tikzcd}$$

where ϑ_Y is the canonical lifting of ϑ to $(Y\times S,0) = (\mathbb{C}^p\times S,0)$. Let $(y_1,\dots,y_p)$ be the coordinates of $\mathbb{C}^p$ so that $\mathcal{O}_{\mathbb{C}^p\times S,0} = \mathcal{O}_{S,0}\{Y_1,\dots,Y_p\}$. Using the commutativity of the square on the right hand side it follows that

$$(1-\vartheta_X)(Y_i\circ F) = ((1-\vartheta_Y)(Y_i))\circ F_\vartheta,$$

whence

$$F_i - \vartheta_X(F_i) = (F_\vartheta)_i$$

as $\vartheta_Y(Y_i)=0$. With the identification above we get $F_\vartheta = F-\varepsilon H$, $H\in\mathcal{M}_X^p$, with $H = -\vartheta_X(F)$.

Finally we remark that by definition $\mathrm{Aut}_G(F/S,\mathcal{M})$ is the set of all pairs (g,ψ) where ψ is an S-automorphism of $S[\mathcal{M}]$ and $g\in G(\mathcal{M})$ is an element with $g.F = \psi^*(F)$. Projecting onto g gives the map α. It is an easy exercise to check that the second square in the above diagram becomes commutative. $\square$

If $\mathcal{M} = \mathcal{O}_{S,0}$, we write simply $TG(F)$ for the image $\gamma_G^F(G(\mathcal{O}_{S,0}))$ in $\mathcal{O}_{X\times S,0}^p$. Let us make the computation of the Kodaira-Spencer class more concrete by the following example.

EXAMPLES 4.2.4. (1) Let $F:(\mathbb{C}\times\mathbb{C}^n,0)\to(\mathbb{C},0)$ be the unfolding

$$F(z,a_0,\dots,a_{n-1}) := z^n + a_{n-1}z^{n-1}+\cdots+a_0$$

of $f(z):=z^n$, $z\in\mathbb{C}$. We want to compute the Kodaira-Spencer class $\delta_{KS}(\partial/\partial a_\nu)$ in $\mathrm{Ex}_G(F/\mathbb{C}^n,\mathcal{O}_{\mathbb{C}^n})$ under the above identification

$$\mathrm{Ex}_G(F/\mathbb{C}^n,\mathcal{O}_{\mathbb{C}^n,0}) \cong \mathcal{O}_{\mathbb{C}\times\mathbb{C}^n,0}/TG(F).$$

For this we have to differentiate F with respect to a_ν. Thus the Kodaira-Spencer class $\delta_{KS}(\partial/\partial a_\nu)$ is the class of z^ν in $\mathcal{O}_{\mathbb{C}\times\mathbb{C}^n,0}/TG(F)$.

(2) As another example, take the unfolding

$$F(z_1,z_2,t) := (z_1^2+tz_2^2, z_1z_2+z_2^3+tz_1)$$

of $f(z_1,z_2) := (z_1^2, z_1z_2+z_2^3)$. Differentiating with respect to t we see that $\delta_{KS}(\partial/\partial t)$ is given by the class of (z_2^2,z_1) in $\mathcal{O}^2_{\mathbb{C}^2\times\mathbb{C},0}/TG(F)$.

We will compute the modules $G(\mathcal{M})$ and the map $\gamma_G = \gamma_G^F$ for the various examples given in 4.1.5.

PROPOSITION 4.2.5. (1) *In the case of extended right equivalence*

$$G(\mathcal{M}) = \mathcal{R}_e(\mathcal{M}) \cong \mathrm{Der}_S(\mathcal{O}_{X\times S,0},\mathcal{M}_X),$$

and the map $\gamma_{\mathcal{R}_e}$ *is given by evaluation on* F*, i.e.*

$$\gamma_{\mathcal{R}_e}(\delta) := (\delta(F_1),\dots,\delta(F_p)) \in \mathcal{M}_X^p.$$

Moreover, $\mathcal{R}(\mathcal{M}) \cong \mathrm{Der}_S(\mathcal{O}_{X\times S,0},\mathfrak{m}_X\mathcal{M}_X)$ *where* $\mathfrak{m}_X$ *denotes the maximal ideal of* $\mathcal{O}_{X,0}$*, and* $\gamma_{\mathcal{R}}$ *is the restriction of* $\gamma_{\mathcal{R}_e}$*.*

(2) *In the case of extended left equivalence we have with* $Y := \mathbb{C}^p$

$$G(\mathcal{M}) = \mathcal{L}_e(\mathcal{M}) \cong \mathcal{M}_Y^p$$

where $\mathcal{M}_Y := \mathcal{O}_{\mathbb{C}^p \times S,0} \otimes_{\mathcal{O}_{S,0}} \mathcal{M}$. *Here* $\gamma_{\mathcal{L}_e}$ *is the map* $-F^* : \mathcal{M}_Y^p \to \mathcal{M}_X^p$. *Again* $\mathcal{L}(\mathcal{M}) \cong \mathfrak{m}_Y \mathcal{M}_Y^p$ *where* $\mathfrak{m}_Y$ *is the maximal ideal of* $\mathcal{O}_{Y,0}$, *and* $\gamma_{\mathcal{R}}$ *is the restriction of* $\gamma_{\mathcal{R}_e}$.

(3) *In the case of* $\mathcal{A}_e = \mathcal{R}_e \times \mathcal{L}_e$*-equivalence* $\mathcal{A}_e(\mathcal{M}) = \mathcal{R}_e(\mathcal{M}) \oplus \mathcal{L}_e(\mathcal{M})$ *and* $\gamma_{\mathcal{A}_e} = \gamma_{\mathcal{R}_e} + \gamma_{\mathcal{L}_e}$. *Similarly,* $\mathcal{A}(\mathcal{M}) = \mathcal{R}(\mathcal{M}) \oplus \mathcal{L}(\mathcal{M})$, *and* $\gamma_{\mathcal{L}}$ *is the restriction of* $\gamma_{\mathcal{L}_e}$.

(4) *In the case of* $\mathcal{C}$*-equivalence*

$$\mathcal{C}(\mathcal{M}) = \mathrm{End}_{X\times S}\left(\mathcal{O}_{X\times S,0}^p\right) \otimes_{\mathcal{O}_{S,0}} \mathcal{M},$$

and $\gamma_{\mathcal{C}}$ *is given through*

$$\mathcal{C}(\mathcal{M}) \ni M \longmapsto -MF \in \mathcal{O}_{X\times S,0}^p \otimes \mathcal{M},$$

which is up to the sign the natural action of endomorphisms on the vector $F = (F_1, \ldots, F_p)$. *In particular,* $\mathrm{Ex}_G(F/S, \mathcal{M}) \cong \mathcal{M}_X^p$, *where* $X = F^{-1}(0)$.

(5) *In the case of* $\mathcal{K}_e$*-equivalence* $\mathcal{K}_e(\mathcal{M}) = \mathcal{C}(\mathcal{M}) \oplus \mathcal{R}_e(\mathcal{M})$, *and* $\gamma_{\mathcal{K}_e} = \gamma_{\mathcal{C}} + \gamma_{\mathcal{R}_e}$. *Similarly,* $\mathcal{K}(\mathcal{M}) = \mathcal{C}(\mathcal{M}) \oplus \mathcal{R}(\mathcal{M})$, *and* $\gamma_{\mathcal{K}}$ *is the restriction of* $\gamma_{\mathcal{K}_e}$.

PROOF. It suffices to prove (1), (2) and (4). In the case (1) an element h of $\mathcal{R}_e(\mathcal{M})$ is just an infinitesimal automorphism of $(X \times S)[\mathcal{M}_X]$ and has therefore the form $h = 1 - \varepsilon\delta$ for some derivation $\delta \in \mathrm{Der}_S(\mathcal{O}_{X\times S,0}, \mathcal{M}_X)$, see **??**. This proves the first part of (1). To compute $\gamma_{\mathcal{R}_e}$ observe that

$$(F \circ (1 - \varepsilon\delta))^*(Y_i) = (1 - \varepsilon\delta)(F_i) = F_i + \varepsilon\delta(F_i)$$

where Y_i denotes the i-th coordinate on $Y = \mathbb{C}^p$. Using the definitions $\gamma_{\mathcal{R}_e}$ has the required form. In the case of $\mathcal{R}$-equivalence an infinitesimal automorphism $1 - \varepsilon\delta$ of $(X \times S[\mathcal{M}], 0)$ must be the identity on $0 \times S[\mathcal{M}]$. Hence $\delta \equiv 0 \bmod \mathfrak{m}_X$, i.e. $\mathcal{R}(\mathcal{M}) = \mathrm{Der}_S(\mathcal{O}_{X\times S,0}, \mathfrak{m}_X\mathcal{M}_X)$.

In the case of (2) of we get with the same argument

$$\mathcal{L}_e(\mathcal{M}) \cong \mathrm{Der}_S(\mathcal{O}_{Y\times S,0}, \mathcal{M}_Y) \cong \mathcal{M}_Y^p.$$

To compute $\gamma_{\mathcal{L}_e}$ write an infinitesimal automorphism of $(Y[\mathcal{M}], 0)$ as $1 - \varepsilon\delta$ with $\delta \in \mathcal{L}_e(\mathcal{M}) \cong \mathrm{Der}_S(\mathcal{O}_{Y\times S,0}, \mathcal{M}_Y)$. We get

$$((1 - \varepsilon\delta) \circ F)^*(Y_i) = F^*(Y_i - \varepsilon\delta(Y_i)) = F_i - \varepsilon F^*(\delta(Y_i)).$$

Under the identification $\mathrm{Der}_S(\mathcal{O}_{Y\times S,0}, \mathcal{M}_Y) \cong \mathcal{M}_Y^p$, $\delta \hat{=} (\delta(Y_i))_{1\le i\le n}$ this means that $\gamma_{\mathcal{L}_e}$ is just the map given by $F^* : \mathcal{M}_Y^p \to \mathcal{M}_X^p$. The case of $\mathcal{L}$-equivalence follows again from the case of $\mathcal{L}_e$-equivalence using the same argument as in the proof of (1).

Finally in the case of (4)

$$\mathcal{C}(S, 0) = GL\left(\mathcal{O}_{X\times S,0}^p\right)$$

and similarly as above

$$\mathcal{C}(\mathcal{M}) = \mathrm{End}\left(\mathcal{O}_{X\times S,0}^p\right) \otimes \mathcal{M} \ .$$

Moreover, if $u = 1 + \varepsilon w \in \mathcal{C}\,(S[\mathcal{M}], 0)$ with $w \in \mathcal{C}(\mathcal{M})$ then

$$(1 + \varepsilon w)F = F + \varepsilon wF,$$

which proves (4). □

Examples 4.2.6. (1) Let us return to the example discussed in 4.2.4 (1). We get for $\mathcal{K}$-equivalence that

$$\mathrm{Ex}_{\mathcal{K}}(F/\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \cong \mathcal{O}_{\mathbb{C}\times\mathbb{C}^n,0}/((F) + \mathfrak{m}_{\mathbb{C}}(\partial F/\partial t)).$$

The classes of $\delta_{KS}(\partial/\partial a_\nu) = z^\nu$ generate this module. Moreover they form a basis of the $\mathbb{C}$-vector space $\mathrm{Ex}_{\mathcal{R}}(F/\mathbb{C}^n, \mathbb{C})$. Using 3.4.17 it follows that F is the formally semiuniversal deformation of $f(z) = z^n$.

(2) In this example let us consider in the case of $\mathcal{A}_e$-equivalence the map

$$f = (f_0, \ldots, f_n) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$$

where $f_\nu = z_\nu$ for $0 \le \nu \le n-1$ and

$$f_n(z_0, \ldots, z_n) := z_n^{n+1} + z_{n-1}z_n^{n-1} + \cdots + z_1 z_n + z_0.$$

Denote by e_ν, $0 \le \nu \le n$, the standard basis of $\mathcal{O}^{n+1}_{\mathbb{C}^{n+1},0}$. The subgroup $T\mathcal{A}_e(f)$ is the sum of $f^*(\mathcal{O}_{\mathbb{C}^{n+1},0})$ together with

$$\sum_{\nu=0}^{n} \partial f/\partial z_\nu \mathcal{O}_{\mathbb{C}^{n+1},0} = (\sum_{\nu=0}^{n-1}(e_\nu + z_n^\nu e_n)\mathcal{O}_{\mathbb{C}^{n+1},0}) + (nz_n^n + \sum_{\nu=0}^{n-1} \nu z_\nu z_n^{\nu-1})e_n\mathcal{O}_{\mathbb{C}^{n+1},0}.$$

Dividing out the second summand gives as quotient

$$Q := \mathcal{O}_{\mathbb{C}^{n+1},0}/(nz_n^n + \sum_{\nu=0}^{n-1} \nu z_\nu z_n^{\nu-1}),$$

and the class of e_ν is identified with $-z_n^\nu$. As these elements generate that module over $f^*(\mathcal{O}_{\mathbb{C}^{n+1},0})$ this proves that $\mathrm{Ex}_{\mathcal{R}_e}(f, \mathbb{C}) = 0$. With other words, f is rigid. In the theory of singularities of mappings, such maps are also called *stable*.

In section 1.2 we investigated integration of vector fields and applied this in the context of finite determinacy of hypersurface singularities. We will now give the promised interpretation of this in terms of Kodaira-Spencer classes which allows to generalize the results of section 1.2 to the case of other groups.

We first consider the general situation.

4.2.7. Let $p : \mathbf{F} \to \mathbf{An}_{(\Sigma,0)}$ be a local deformation theory. For $a \in F(S, 0)$ with a germ $(S, 0)$ assume that there is an infinitesimal automorphism $\alpha \in \mathrm{Aut}(a, \mathcal{O}_{S,0})$ such that the image, say δ, of α in

$$\mathrm{Der}_\Sigma(\mathcal{O}_{S,0}, \mathcal{O}_{S,0})$$

is a derivation with $\delta(t) = 1$ for some $t \in \mathfrak{m}_{S,0}$. According to 2.1.5 we obtain a decomposition $(S, 0) \cong (S_0 \times \mathbb{C}, 0)$ by integrating δ. We will say that a can be *trivialized along* δ if there exists an isomorphism

$$a \cong pr^*(a|S_0)$$

where $pr : S \cong S_0 \times \mathbb{C} \to S_0$ is the projection.

Definition 4.2.8. A deformation theory is said to admit *integration of vector fields* if in the above situation every objects $a \in F(S)$ for which there is an infinitesimal automorphism α as above can be trivialized along δ.

Let $a \in \mathbf{F}(\mathbb{C}, 0)$ be a deformation over $(\mathbb{C}, 0)$. In view of the Kodaira-Spencer sequence 3.3.4

$$\operatorname{Aut}(a, \mathcal{O}_{\mathbb{C},0}) \longrightarrow \operatorname{Der}(\mathcal{O}_{\mathbb{C},0}, \mathcal{O}_{\mathbb{C},0}) \xrightarrow{\delta_{KS}} \operatorname{Ex}(a/S, \mathcal{O}_{\mathbb{C},0})$$

the integrability of a vector field is decided by the Kodaira-Spencer class, namely:

THEOREM 4.2.9. *Let $p : \mathbf{F} \to \mathbf{An}_{\Sigma,0)}$ be a deformation theory admitting integration of vector fields. Let $a \in \mathbf{F}(\mathbb{C}, 0)$ be a deformation of a_0. Then a is trivial iff the Kodaira-Spencer class $\delta_{KS}(\partial/\partial t) \in \operatorname{Ex}(a/S, \mathcal{O}_{\mathbb{C},0})$ vanishes.*

PROPOSITION 4.2.10. *Let G be one of the groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$, $\mathcal{K}$ or one of the associated extended groups. Then the corresponding deformation theory $p : \mathbf{E}_G \to$* **Germs** *admits integration of vector fields.*

PROOF. Let us treat the case that $G = \mathcal{R}$. First observe that an element of $\operatorname{Aut}_{\mathbf{E}_G}(F, \mathcal{O}_S)$ is a pair (g, ψ) where ψ is an S-automorphism of $(S[\mathcal{O}_S], 0)$ and $g \in G(\mathcal{O}_S)$ is an element with $g.F = \psi^*(F)$. This element maps to $\delta \in \operatorname{Der}(\mathcal{O}_{S,0}, \mathcal{O}_{S,0})$ if $\psi = 1 - \varepsilon\delta$. Moreover g is an X-automorphism of $(X[\mathcal{O}_X], 0)$ and has therefore the form $1 - \tilde{\delta}$ for some derivation $\tilde{\delta} \in \operatorname{Der}(\mathcal{O}_{X\times S,0}, \mathcal{O}_{X\times S,0})$. The equation $g.F = \psi^*(F)$ just means that $(\tilde{\delta}, \delta)$ is a pair of compatible derivations, i.e. the diagram

$$\begin{tikzcd}
\mathcal{O}_{S,0} \arrow[r] \arrow[d, "\delta"'] & \mathcal{O}_{Y\times S,0} \arrow[r, "F^*"] \arrow[d, "proj^*(\delta)"'] & \mathcal{O}_{X\times S,0} \arrow[d, "\tilde{\delta}"'] \\
\mathcal{O}_{S,0} \arrow[r] & \mathcal{O}_{Y\times S,0} \arrow[r, "F^*"] & \mathcal{O}_{X\times S,0}
\end{tikzcd}$$

commutes.

Now assume that $\delta(t) = 1$ for some element $t \in \mathfrak{m}_{S,0}$. Applying 2.1.7 we obtain that there are isomorphisms $h : (S, 0) \to (S_0 \times \mathbb{C}, 0)$ and $\tilde{h} : (X \times S, 0) \to (X \times S_0 \times \mathbb{C}, 0)$ which are compatible, i.e. $(h \times id_Y) \circ F = F \circ \tilde{h}$.

The proof in the case of $\mathcal{L}$- and $\mathcal{A}$-equivalence is the same.

In the case of $\mathcal{C}$-equivalence observe that $G(S, 0)$ can be identified with the set of all S-automorphisms Φ of $(X \times Y \times S, 0)$ of the form

$$\Phi(x, y, s) = (x, \varphi(x, y, s), s)$$

where $\varphi(x, y, s)$ is linear with respect to y. Thus we can again apply **??** as above to obtain integration of vector fields for $\mathcal{C}$-equivalence. The case of $\mathcal{K}_{(e)}$-equivalence now follows similarly by interpreting the elements of $\mathcal{K}_{(e)}(S, 0)$ as S-automorphisms Φ of $(X \times Y \times S, 0)$ of type

$$\Phi(x, y, s) = (\Phi_1(x, s), \varphi(x, y, s), s)$$

where $(\Phi_1(x, s), s)$ gives an automorphism of $(X \times S, 0)$ and $\varphi(x, y, s)$ is linear in y. This concludes the proof. □

EXERCISE 4.2.11. Show that the following maps are rigid with respect to $\mathcal{A}_e$-equivalence:

$$f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad (z_1, \dots, z_n) \mapsto z_1^2 + \cdots + z_n^2;$$
$$f : (\mathbb{C}^4, 0) \to (\mathbb{C}^4, 0), \quad (z_1, \dots, z_4) \mapsto (z_1, z_2, z_3^2 + z_1 z_4, z_4^2 + z_2 z_3);$$
$$f : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0), \quad (z_1, z_2, z_3) \mapsto (z_1, z_2^2 + z_3^3 + z_1 z_3).$$

4.3. Finite Determinacy

We start by the following immediate application of 4.2.11 which is our basic criterion.

PROPOSITION 4.3.1. *Let G be one of the groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$, $\mathcal{K}$ or one of the associated extended groups. Let $F \in E(\mathbb{C},0) \cong \mathfrak{m}^p_{X\times\mathbb{C},0}$ be a 1-parameter unfoldings of $f : (X,0) \to (\mathbb{C}^p,0)$. Then the following are equivalent.*

1. *F is G-equivalent to $f_{\mathbb{C}} : (X \times \mathbb{C}, 0) \to (\mathbb{C}^p \times \mathbb{C}, 0)$.*
2. *$\partial F/\partial t \in TG(F)$.*

PROOF. This follows immediately from 4.2.11 and the fact that the Kodaira-Spencer class $\delta_{KS}(\partial/\partial t)$ is given by the class of $\partial F/\partial t$ in

$$\mathrm{Ex}_G(F/\mathbb{C}, \mathcal{O}_{\mathbb{C},0}) \cong \mathcal{O}^p_{X\times\mathbb{C},0}/TG(F),$$

see 4.2.3. □

Consider in particular an unfolding of the form

$$F_h(x,t) := f(x) + h(t)g(x) \in E(\mathbb{C},0)$$

with $f, g : (X,0) \to (\mathbb{C}^p,0)$ mapping germs and $h \in \mathcal{O}_{\mathbb{C},0}$ a holomorphic function. Applying 4.3.1 yields the following more technical criterion.

LEMMA 4.3.2. *Let G be one of the groups considered in 4.3.1. Assume that $h(0) = 0$ and that*

1. *$TG(F_h) \supseteq TG(f_{\mathbb{C}})$ in $\mathcal{O}^p_{X\times\mathbb{C},0}$,*
2. *$g \in TG(f)$.*

Then $f \times id_{\mathbb{C}}$ and F_h are G-equivalent.

PROOF. There is a canonical map from $TG(f)$ into $TG(f_{\mathbb{C}})$. Thus from (1) and (2) we obtain that the Kodaira-Spencer class $\delta_{KS}(\partial/\partial t) = h'(t)g$ of F_h vanishes in $\mathrm{Ex}_G(F_h/S, \mathcal{O}_{\mathbb{C},0})$. As $F_h(x,0) = f(x)$, the result follows from 4.3.1 □

LEMMA 4.3.3. *Let G be one of the groups $\mathcal{R}, \mathcal{L}, \mathcal{A}, \mathcal{C}, \mathcal{K}$. Assume that for every function $h \in \mathcal{O}_{\mathbb{C},0}$ we have*

1. *$TG(F_h) \supseteq TG(f_{\mathbb{C}})$ in $\mathcal{O}^p_{X\times\mathbb{C},0}$,*
2. *$g \in TG(f)$.*

Then f and $f + g$ are G-equivalent.

PROOF. Set $F_{t_0} := f + t_0 g$, $F_{t_0+t} := f + (t + t_0)g$, considered as germs in $\mathcal{O}^p_{X\times\mathbb{C},0}$. It suffices to show that F_{t_0} and F_{t_0+t} are G-equivalent for every $t_0 \in \mathbb{C}$. For the (not extended) groups G under consideration, each $\gamma \in G(\mathbb{C},0)$ satisfies $\gamma^*(\mathfrak{m}_X \mathcal{O}^p_{X\times\mathbb{C},0}) \subseteq \mathfrak{m}_X \mathcal{O}^p_{X\times\mathbb{C},0}$, so that the section $0 \times \mathbb{C}$ in $X \times \mathbb{C}$ is preserved. By our assumption (1)

$$TG(F_{t_0+t}) \supseteq TG(f_{\mathbb{C}}).$$

Using again that $TG(f)$ is contained in $TG(f_{\mathbb{C}})$, assumption (2) implies the desired result as in the proof of 4.3.2. □

Applying this to $\mathcal{R}$-equivalence gives the following explicit criterion.

COROLLARY 4.3.4. *Let $f, g \in \mathfrak{m}_X \mathcal{O}^p_{X,0}$ be p-tuples of functions on $(X,0)$. Assume that $T\mathcal{R}(g) \subseteq \mathfrak{m}_X T\mathcal{R}(f)$ and $g \in T\mathcal{R}(f)$. Then f and $f+g$ are $\mathcal{R}$-equivalent.*

Proof. By 4.3.3 it is sufficient to prove that for $h \in \mathcal{O}_{\mathbb{C},0}$, $F := f + h(t)g$

(a) $T\mathcal{R}(F) = T\mathcal{R}(f \times id_{\mathbb{C}})$.

(b) $g \in T\mathcal{R}(f)$.

Clearly (b) follows immediately from our assumption. For the proof of (a) observe that the left hand side is generated by the elements $\vartheta(F), \vartheta \in \mathrm{Der}_S(\mathcal{O}_{X\times\mathbb{C}}, \mathfrak{m}_X \mathcal{O}_{X\times\mathbb{C},0})$. By assumption

$$\vartheta(F) - \vartheta(f) = h(t)\vartheta(g)$$

lie in $\mathfrak{m}_X T\mathcal{R}(f_{\mathbb{C}}) \subseteq T\mathcal{R}(f \times id_{\mathbb{C}})$. Thus (a) follows from Nakayama's lemma. ☐

Let us specialize this to the case of functions $f, g : (X, 0) \to (\mathbb{C}, 0)$. Here $T\mathcal{R}_e(f)$ is the ideal in $\mathcal{O}_{X,0}$ of all $\delta(f)$ with $\delta \in \mathrm{Der}(\mathcal{O}_{X,0}, \mathcal{O}_{X,0})$. In analogy with the case $X = \mathbb{C}^N$ (see 2.1.8) we call this ideal the *Jacobian ideal* of $(X, 0)$ and denote it by $\mathrm{jac}(f)$, i.e. $T\mathcal{R}_e(f) = \mathrm{jac}(f)$ and $T\mathcal{R}(f) = \mathfrak{m}_X \mathrm{jac}(f)$. Hence we get

Corollary 4.3.5. *If* $\mathfrak{m}_X \mathrm{jac}(g) \subseteq \mathfrak{m}^2 \mathrm{jac}(f)$ *and* $g \in \mathfrak{m}_X \mathrm{jac}(f)$ *then* f *and* $f + g$ *are* $\mathcal{R}$*-equivalent.*

A similar result holds for $\mathcal{K}$-equivalence.

Corollary 4.3.6. *Let* $f, g \in \mathfrak{m}_X \mathcal{O}^p_{X,0}$ *be p-tuples of functions on* $(X, 0)$*. If* $T\mathcal{K}(g) \subseteq \mathfrak{m}_X T\mathcal{K}(f)$ *then* f *and* $f + g$ *are* $\mathcal{K}$*-equivalent.*

Proof. With the notations of the proof of 4.3.4, we need to show that that (a) $T\mathcal{K}(F) = T\mathcal{K}(f \times id_{\mathbb{C}})$, and (b) $g \in T\mathcal{K}(f)$. Since g lies in $T\mathcal{K}(g)$ (b) is a consequence of the assumption. Using the lemma of Nakayama as in the proof of 4.3.4 also (a) follows. ☐

Comparing **??** and 4.2.5 (5) we get for an unfolding $F : (\mathbb{C}^n \times S, 0) \to (\mathbb{C}^p, 0)$ that the modules $T\mathcal{K}(F)$ and $\sum F_i \mathcal{O}^p_{X\times S,0} + \mathfrak{m}_X \mathrm{jac}_S(F)$ of $\mathcal{O}^p_{X\times S,0}$ coincide. Therefore 4.3.5 is an improvement of 2.2.2. We also note the case of a function $f : (X, 0) \to (\mathbb{C}, 0)$. Here $T\mathcal{K}_e(f)$ is the ideal generated by f and $\mathrm{jac}(f)$, and $T\mathcal{K}(f) = (f) + \mathfrak{m}_X \mathrm{jac}(f)$, and we get:

Corollary 4.3.7. *Let* $f, g : (X, 0) \to (\mathbb{C}, 0)$ *be functions. If* $(g) + \mathfrak{m}_X \mathrm{jac}(g) \subseteq \mathfrak{m}(g) + \mathfrak{m}^2_X T\mathcal{K}(f)$ *then* f *and* $f + g$ *are* $\mathcal{K}$*-equivalent.*

Remarks 4.3.8. (1) Similarly, f, g are $\mathcal{C}$-equivalent if $T\mathcal{C}(g) \subseteq \mathfrak{m}_X T\mathcal{C}(f)$, i.e. if $g_i \in \mathfrak{m}_X \cdot \sum_j f_j \mathcal{O}_{X,0}$ for all i.

(2) In case of a homogeneous function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ the module $\mathfrak{m}\,\mathrm{jac}(f)$ contains f. Hence, if for an arbitrary function $g \in \mathcal{O}_{\mathbb{C}^N,0}$ we have that $(*)$ $\mathfrak{m}\,\mathrm{jac}(g) \subseteq \mathfrak{m}^2_X \mathrm{jac}(f)$ then f and $f + g$ are $\mathcal{R}$-equivalent. In fact, if $(*)$ holds then it is also satisfied for every homogeneous component, say g_ρ, of g. By the Euler identity we have $g_\rho \in \mathfrak{m}^2_X \mathrm{jac}(f)$, and so $g \in \mathfrak{m}^2_X \mathrm{jac}(f)$.

We now turn — similarly as in sect. 1.2 — to finite determinacy. This notion can be introduced for arbitrary groups acting on map germs.

Definition 4.3.9. Let G be a homogeneous group valued functor acting on E. A map germ $f : (X, 0) \to (\mathbb{C}^p, 0)$ is called *r-G-determined* if for every map germ $g : (X, 0) \to (\mathbb{C}^p, 0)$ with $g \equiv f \bmod \mathfrak{m}^{r+1}_{X,0}$ the germs f and g are G-equivalent. Moreover, f is called finitely G-determined if it is r-G-determined for some $r \in \mathbb{N}$.

In the smooth case $(X, 0) = (\mathbb{C}^n, 0)$, if $f \in \mathcal{O}^p_{X,0}$ and f_r denotes its r-jet, i.e. its Taylor expansion up to order r, then f is r-G-determined iff f_r is r-G-determined. In particular, if this is the case then f and f_r are G-equivalent, i.e. f is G-equivalent to a polynomial map.

THEOREM 4.3.10. *Let $f : (X, 0) \to (\mathbb{C}^p, 0)$ be a holomorphic map germ and G one of the groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{C}$, $\mathcal{K}$, $\mathcal{A}$. We set $\varepsilon = \varepsilon(G) = 1$ if $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, and $\varepsilon = \varepsilon(G) = 2$ for $G = \mathcal{L}$ or $\mathcal{A}$. If*

$$\mathfrak{m}_X^{r+1}\mathcal{O}^p_{X,0} \subseteq TG(f)$$

then f is $(\varepsilon r + 1)$-G-determined.

PROOF. The claim is that f and $f + g$ are G-equivalent for every $g \in m_X^{\varepsilon r+2}$. For the cases $G = \mathcal{R}, \mathcal{K}, \mathcal{C}$ this is just 4.3.4, 4.3.6 and 4.3.8. Now assume that $G = \mathcal{L}$. By 4.3.3 it is sufficient to show that for $h \in \mathcal{O}_{\mathbb{C},0}$ and $F := f + h(t)g$

(a) $T\mathcal{L}(F) = T\mathcal{L}(f_{\mathbb{C}})$,

(b) $g \in T\mathcal{L}(f)$.

By assumption, (b) is satisfied. For (a) consider the Taylor expansion

$$\begin{aligned} F^*(\varphi) - (f_{\mathbb{C}})^*(\varphi) &= \varphi(f + hg) - \varphi(f) \\ &= \sum_{\substack{i \neq 0 \\ i \in \mathbb{N}^p}} \frac{1}{i!} \frac{\partial^{|i|}\varphi}{\partial y^i}(f) \cdot (hg)^i, \end{aligned}$$

where $\varphi \in \mathfrak{m}_{\mathbb{C}^p,0}$ and $y_1, \ldots, y_p$ are the coordinates of $\mathbb{C}^p$. This expansion shows that $F^*(\varphi) - (f_{\mathbb{C}})^*(\varphi)$ is contained in $\mathfrak{m}_X^{2r+2}M$, where $M := \mathcal{O}^p_{X\times\mathbb{C},0}$. Thus with $I := \mathfrak{m}_X^{r+1}$ we get

$$T\mathcal{L}(F) + I^2M = T\mathcal{L}(f_{\mathbb{C}}) + I^2M,$$

and by assumption IM is contained in $T\mathcal{L}(f_{\mathbb{C}})$. By the lemma below $IM \subseteq T\mathcal{L}(F)$ and so (a) follows.

Finally, in the case $G = \mathcal{A}$ we again have to show that (a) and (b) above hold with $T\mathcal{L}$ replaced by $T\mathcal{A}$. With the same arguments we get that

$$T\mathcal{L}(F) + T\mathcal{R}(F) + I^2M = T\mathcal{L}(f_{\mathbb{C}}) + T\mathcal{R}(f_{\mathbb{C}}) + I^2M$$

and that IM is contained in $T\mathcal{L}(f_{\mathbb{C}}) + T\mathcal{R}(f_{\mathbb{C}})$. Applying the lemma below to $\bar{M} := M/T\mathcal{R}(F)$ and the image, say $\bar{N}$, of $T\mathcal{L}(F)$ in $\bar{M}$ we obtain that $I\bar{M} \subseteq \bar{N}$. Hence $IM \subseteq T\mathcal{L}(F) + T\mathcal{R}(F)$ which gives (a). □

The following lemma is a variant of Nakayama's lemma which was shown by du Plessis [**Ple**].

LEMMA 4.3.11. *Let $(A, \mathfrak{n}) \to (B, \mathfrak{m})$ be a homomorphism of analytic algebras, M a finitely generated B-module and $N \subseteq \mathfrak{n}M$ an A-submodule which is finite over A. Assume that $I \subseteq \mathfrak{m}$ is an ideal in B such that*

$$(*) \qquad IM \subseteq N + I^2M.$$

Then $IM \subseteq N$.

PROOF. The hypothesis implies $IM \subseteq BN + I^2M$. By Nakayama's lemma $IM \subseteq BN$. Hence $I^2M \subseteq \mathfrak{n}IM$. Substituting into $(*)$ gives

$$(**) \qquad IM \subseteq N + \mathfrak{n}IM.$$

In particular $IM/\mathfrak{n}IM$ is a finite A-module. By the preparation theorem **??** IM is finite over A. Applying Nakayama's lemma to the inclusion of finite A-modules $(**)$ we get $IM \subseteq N$. □

Remark 4.3.12. There is also a converse to 4.3.10. If f is k-determined then $\mathfrak{m}_X^{k+1}\mathcal{O}_{X,0}^p \subseteq TG(f)$. For a proof we refer the reader to [**Wal**, Theorem 1.2 (i)]

Remark 4.3.13. The above results hold m.m. for unfoldings, with almost the same proofs! For instance, *let* $F : (X \times S, 0) \to (\mathbb{C}^p, 0)$ *be an unfolding of* $f(z) := F(z, 0)$ *and let* G, $\varepsilon = \varepsilon(G)$ *be as in 4.3.10. If*

$$\mathfrak{m}_X^{r+1}\mathcal{O}_{X\times S,0}^p \subseteq TG(F)$$

then F *is* $(\varepsilon r + 1)$*-*G*-determined, i.e. every map* $\tilde{F}$ *with* $\tilde{F} - F \in \mathfrak{m}_X^{\varepsilon r+2}$ *is* G*-equivalent to* F.See [**BFl**] for a proof.

4.4. Theorems of Mather-Yau Type

Let $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$ functions defining hypersurfaces H_f and H_g in a neighbourhood of 0. If $(H_f, 0)$ and $(H_g, 0)$ are isomorphic then also their Jacobian algebras $\mathcal{O}_{\mathbb{C}^n,0}/\operatorname{jac}^e(f)$ and $\mathcal{O}_{\mathbb{C}^n,0}/\operatorname{jac}^e(g)$ are isomorphic. In this section we will give a proof of the surprising fact that also the converse holds provided that the singularities are isolated. More precisey, we will show that the following result holds.

Theorem 4.4.1. (Mather-Yau) *Let* $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$ *be functions and assume that* f *defines an isolated hypersurface singularity. If the Jacobian algebras*

$$\mathcal{O}_{\mathbb{C}^n,0}/\operatorname{jac}^e(f) \quad \textit{and} \quad \mathcal{O}_{\mathbb{C}^n,0}/\operatorname{jac}^e(g)$$

are isomorphic then f *and* g *define isomporphic hypersurface singularities.*

This theorem will be deduced as a special case of a more general result, see **??** below. We also give a similar result for the $\mathcal{A}$-equivalence. We will use the following notations.

4.4.2. Let $(X, 0)$ be a germ of a complex space. By $\mathfrak{m}_X\mathcal{O}_{X,0}$ we denote the maximal ideal. For a p-tuple of functions f on $(X, 0)$ and a germ $(S, 0)$ we will denote by f_S the map $f_S(x, s) := f(x)$. If F is an unfolding of f over the germ $(S, 0)$ then by $J_F : \operatorname{Der}_S(\mathcal{O}_{X\times S,0}, \mathcal{O}_{X\times S,0}) \to \mathcal{O}_{X\times S,0}^p$ we indicate the map $\delta \mapsto \delta(F)$.

We start with the following variant of 4.3.6.

Lemma 4.4.3. *Let* $f, g \in \mathfrak{m}_X\mathcal{O}_{X,0}^p$ *be* p*-tuples of functions on* $(X, 0)$*. If* $T\mathcal{K}(g) \subseteq T\mathcal{K}(f)$ *then for* ε *sufficiently small the germs* f *and* $f + \varepsilon g$ *are* $\mathcal{K}$*-equivalent. The same holds if* $\mathcal{K}$*-equivalence is replaced by* $\mathcal{K}_e$*-equivalence.*

Proof. Consider the unfolding $F := f + tg$ of f, i.e. $F \in \mathcal{O}_{X\times\mathbb{C},0}$. We need to show that F and $f_{\mathbb{C}}$ are $\mathcal{K}$-equivalent. By the criterion 4.3.2 we must verify that

(1) $T\mathcal{K}(F) \supseteq T\mathcal{K}(f_{\mathbb{C}})$.
(2) $g \in T\mathcal{K}(f)$.

By our assumption, (2) holds. For the proof of (1) observe that

$$T\mathcal{K}(F_{\mathbb{C}}) + tT\mathcal{K}(g_{\mathbb{C}}) = T\mathcal{K}(f_{\mathbb{C}}) + tT\mathcal{K}(g_{\mathbb{C}}).$$

As $T\mathcal{K}(g) \subseteq T\mathcal{K}(f)$ the lemma of Nakayama gives (1). □

In the case of $\mathcal{K}$-equivalence the most general Mather-Yau type theorem is the following result, see [**GHa**], or [**BFl**, 5.1]

THEOREM 4.4.4. *Let* $f, g \in \mathfrak{m}_X\mathcal{O}^p_{X,0}$ *mapping germs with* $T\mathcal{K}(f) = T\mathcal{K}(g)$ *in* $\mathcal{O}^p_{X,0}$. *Then* f *and* g *are* $\mathcal{K}$*-equivalent.*

PROOF. Consider the mapping germ $F = (1-t)f + tg$, which is defined in some neighbourhood, say U, of $0 \times \mathbb{C}$ in $X \times \mathbb{C}$. By 4.4.3 the mapping germ F is $\mathcal{K}$-equivalent to $f_{\mathbb{C}}(x,t) := f(x)$ resp. $g_{\mathbb{C}}$ in a neighbourhood of $(0,0)$ resp. $(0,1)$ in $X \times \mathbb{C}$. Thus it suffices to show that the (open) set V of $t_0 \in \mathbb{C}$ such that F is $\mathcal{K}$-equivalent to $F_{t_0} := (1-t_0)f + t_0 g$ in a neighbourhood of $(0,t_0)$ in $X \times \mathbb{C}$, is connected. We claim that this set is even Zariski-open. Consider the homomorphism of sheaves $J_F : \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_U, \mathfrak{m}_X\mathcal{O}_U) \to \mathcal{O}^p_U$, which is given by $\delta \mapsto \delta(F)$. The sheaf

$$\mathcal{E}x_{\mathcal{K}}(F) = \mathcal{O}^p_U/(\operatorname{Im}(J_F) + \sum F_i\mathcal{O}^p_U),$$

is coherent on U. The Kodaira-Spencer class $\delta_{KS}(\partial/\partial t)$ may be regarded as a section in $\mathcal{E}x_{\mathcal{K}}(F)$. The set V is just the set of points t_0 such that the germ of $\delta_{KS}(\partial/\partial t)$ at $(0,t_0)$ vanishes, see 4.3.1. As $\mathcal{E}x_{\mathcal{K}}(F)$ is coherent, the support of $\delta_{KS}(\partial/\partial t)$ is a Zariski-closed subset and so V is Zariski-open. ☐

For the extended contact equivalence one can prove a similar result provided that f defines an isolated singularity.

THEOREM 4.4.5. *Assume that* $f, g \in \mathfrak{m}^2_X\mathcal{O}^p_{X,0}$ *are mapping germs such that* $Z := f^{-1}(0)$ *has an isolated singularity at 0. Then* $T\mathcal{K}_e(f) = T\mathcal{K}_e(g)$ *implies that* f *and* g *are* $\mathcal{K}$*-equivalent.*

PROOF. Set again $F := (1-t)f + tg$. Let V be the open set of all $t_0 \in \mathbb{C}$ such that F is $\mathcal{K}$-equivalent to F_{t_0} in a neighbourhood of $(0,t_0)$. The argument of the proof of 4.4.4 shows that V is Zariski-open. We will show that it contains $0 \in \mathbb{C}$. In fact, by 4.4.3, applied to f and $H = f + h(t)g$ with $h(t) = t/(1-t)$, we get that H and $f_{\mathbb{C}}$ are $\mathcal{K}_e$-equivalent in a neighbourhood of $(0,0) \in X \times \mathbb{C}$. This equivalence induces in particular an $\mathbb{C}$-isomorphism α from $Z \times \mathbb{C}$ onto $\mathcal{Z} := H^{-1}(0)$ near $(0,0)$. As $f, g \in \mathfrak{m}^2_X\mathcal{O}^p_{X,0}$, the set $0 \times \mathbb{C}$ is contained in $\operatorname{Sing}(\mathcal{Z})$. Moreover, under the isomorphism α^{-1} the set $\operatorname{Sing}(\mathcal{Z})$ is mapped onto $\operatorname{Sing}(Z) \times \mathbb{C} = 0 \times \mathbb{C}$ so that $0 \times \mathbb{C}$ is preserved, and the given equivalence is already a $\mathcal{K}$-equivalence.

Similarly, applying 4.4.3 to g and $G = g + \tilde{h}(t)f$ with $\tilde{h} = (1-t)/t$ around $t = 1$, we get that G and $g_{\mathbb{C}}$ are $\mathcal{K}_e$-equivalent in a neighbourhood of $(0,1) \in X \times \mathbb{C}$. As V is Zariski-open the mapping germ $G_{t_0}(x) := G(x,t_0)$ is equivalent to f for t_0 near 1, $t_0 \neq 1$. In particular $\operatorname{Sing} G^{-1}_{t_0}(0) = 0$ for such t_0. Moreover, $0 \in \operatorname{Sing}(g^{-1}(0))$ as $g \in \mathfrak{m}^2_X\mathcal{O}^p_X$. It follows that the $\mathcal{K}_e$-equivalence between G and $g_{\mathbb{C}}$ preserves $(0,t_0)$, for $t_0 \neq 1$, t_0 near 1, and hence also $0 \times \mathbb{C}$ in a neighbourhood of $(0,1)$. ☐

When we specialize this to the case of complete intersections we get the following more explicit result.

COROLLARY 4.4.6. (see [**GHa**] ,2. (5)) *Let* $(X,0)$ *and* $(Y,0)$ *be complete intersections of codimension* p *with isolated singularities in* $\mathbb{C}^n$ *which are given by* p*-tuples of functions* $f := (f_1, \ldots, f_p)$ *resp.* $g = (g_1, \ldots, g_p)$. *Assume that*

$$T_{X,0} := \mathcal{O}^p_{\mathbb{C}^n,0}/\operatorname{jac}^e(f) \quad \textit{and} \quad T_{Y,0} := \mathcal{O}^p_{\mathbb{C}^n,0}/\operatorname{jac}^e(g)$$

are isomorphic in the sense that there is an automorphism Φ of $\mathbb{C}^n$ such that these modules are isomorphic as $\mathcal{O}_{\mathbb{C}^n,0}$-modules. Then $(X,0)$ and $(Y,0)$ are isomorphic.

Proof. We may suppose that $T_{X,0} \cong T_{Y,0}$ as $\mathcal{O}_{\mathbb{C}^n,0}$ modules. Then $T_{X,0} = \mathcal{O}^p_{\mathbb{C}^n,0}/T\mathcal{K}_e(f,\mathbb{C})$ and $T_{Y,0} = \mathcal{O}^p_{\mathbb{C},0}/T\mathcal{K}_e(g,\mathbb{C})$. After passing to another minimal system of generators of the ideal of Y we may suppose that the isomorphism $T_{X,0} \cong T_{Y,0}$ is given by the identity on $\mathcal{O}^p_{\mathbb{C}^n,0}$. Now 4.4.5 implies the desired result. □

proof of 4.4.1. In the case of of functions, i.e. $p = 1$, the modules $T_{X,0}$, $T_{Y,0}$ in **??** are just the Jacobian algebras of the functions f resp. g. Hence an isomorphism in the sense of **??** is up to a constant an algebra isomorphism of the Jacobian algebras. Thus 4.4.1 follows. □

Remark 4.4.7. (1) The result 4.4.3 holds for $\mathcal{R}$–equivalence as well — by the same proof — provided that $f \in T\mathcal{R}(f,\mathcal{O}_S)$ and $g \in T\mathcal{R}(g,\mathcal{O}_S)$. For S a point and $p = 1$ this is the case if f and g are quasihomogeneous.

Moreover, if we assume that $f^{-1}(0)$ has singularity set 0 and f, g are in $\mathfrak{m}^2_X\mathcal{O}^p_X$, then $\mathcal{R}$ can be replaced by $\mathcal{R}_e$. This is easily seen by the same argument as in 4.4.4. For similar results see also [**GHa**, 2.(8)] and [**Sho**].

In the remaining part of this section we will prove a theorem of Mather-Yau type for $\mathcal{A}$-equivalence. First we show the following local result.

Proposition 4.4.8. *Let $f, g \in \mathfrak{m}_X\mathcal{O}^p_{X,0}$ be mapping germs such that $\mathrm{Ex}_{\mathcal{K}}(f,\mathbb{C})$ is a finite dimensional vector space over $\mathbb{C}$ and $T\mathcal{A}(g) \subseteq T\mathcal{A}(f)$. Then the mapping germs $f_{\mathbb{C}}(x,t) := f(x)$ and $F := f + tg$ in $\mathcal{O}^p_{X\times\mathbb{C},0}$ are $\mathcal{A}$-equivalent.*

Proof. We set $A := \mathcal{O}_{\mathbb{C}^p\times\mathbb{C},0}$, $B := \mathcal{O}_{X\times\mathbb{C},0}$ and $\bar{A} := \mathcal{O}_{\mathbb{C}^p,0} = A/tA$, $\bar{B} = B/tB$, where t is the coordinate function on $\mathbb{C}$. Let $J_F : \mathrm{Der}_{\mathbb{C}}(B, \mathfrak{m}_X B) \to B^p$ be the canonical map. As $\mathrm{Der}_{\mathbb{C}}(B,\mathfrak{m}_X B) \cong \mathrm{Der}(\bar{B},\mathfrak{m}_X\bar{B}) \otimes_{\mathcal{O}_{X,0}} B$ we have

$$\begin{aligned}\mathrm{Ex}_{\mathcal{K}}(f,\mathbb{C}) &= \bar{B}^p/\operatorname{Im}(J_F \otimes_B \bar{B}) + \sum f_i\bar{B}^p)\\ &= (B^p/\mathrm{Im}\ (J_F)) \otimes_A (A/(\mathfrak{m}_{\mathbb{C}^p}A + tA)).\end{aligned}$$

By assumption this is a finite dimensional vector space over $\mathbb{C} \cong A/(\mathfrak{m}_{\mathbb{C}^p}A + tA)$. The preparation theorem gives that $B^p/\operatorname{Im}(J_F)$ is a finite $F^*(A)$-module. It follows that

$$\mathrm{Ex}_{\mathcal{A}}(F/\mathbb{C},\mathcal{O}_{\mathbb{C},0}) = B^p/(\operatorname{Im}(J_F) + F^*(\mathfrak{m}_{\mathbb{C}^p}A^p))$$

is finite over $F^*(A)$. The same argument gives that $\mathrm{Ex}_{\mathcal{A}}(f_{\mathbb{C}}/\mathbb{C},\mathcal{O}_{\mathbb{C},0})$ is finite over $(f \times id_{\mathbb{C}})^*(A)$. For $\varphi \in A^p$ consider again the Taylor expansion

$$(F, id_{\mathbb{C}})^*(\varphi) - (f \times id_{\mathbb{C}})^*(\varphi) = \sum_{\substack{i\neq 0\\ i\in\mathbb{N}^p}} \frac{1}{i!}(f \times id_{\mathbb{C}})^*\left(\frac{\partial^{|i|}\varphi}{\partial y^i}\right)(tg)^i.$$

Every term in this series is zero in the finite $(f \times id_{\mathbb{C}})^*(A)$-module $\mathrm{Ex}_{\mathcal{A}}(f_{\mathbb{C}}/\mathbb{C},\mathcal{O}_{\mathbb{C},0})$ by our assumption, and so $(F, id_{\mathbb{C}})^*(\varphi) \equiv 0 \bmod T\mathcal{A}(f_{\mathbb{C}},\mathcal{O}_{\mathbb{C},0})$. Thus

$$T\mathcal{A}(F,\mathcal{O}_{\mathbb{C},0}) \subseteq T\mathcal{A}(f_{\mathbb{C}},\mathcal{O}_{\mathbb{C},0}),$$

and the quotient, say Q, satisfies $Q = tQ$ as both restrict to $T\mathcal{A}(f,\mathbb{C})$ modulo t. As Q is a $\mathbb{C}\{t\}$-submodule of the finite $F^*(A)$-module $\mathrm{Ex}_{\mathcal{A}}(F/\mathbb{C},\mathcal{O}_{\mathbb{C},0})$, it is t-adically separated and so $Q = 0$, whence

$$T\mathcal{A}(F) = T\mathcal{A}(f_{\mathbb{C}}).$$

Now the desired result follows from 4.3.2. □

THEOREM 4.4.9. *Let* $f, g \in \mathfrak{m}_X \mathcal{O}^p_{X,0}$ *be mapping germs such that* $\mathrm{Ex}_{\mathcal{K}}(f, \mathbb{C})$ *is a vector space of finite dimension. If the subsets* $T\mathcal{A}(f)$ *and* $T\mathcal{A}(g)$ *of* $\mathcal{O}^p_{X,0}$ *are equal then* f *and* g *are* $\mathcal{A}$*-equivalent.*

PROOF. As in the proof of 4.4.4 we consider the unfolding $F := (1-t)f + tg$. Since $T\mathcal{K}(f)$ is the $\mathcal{O}_{X,0}$-submodule generated by $T\mathcal{A}(f)$ it follows that $\mathrm{Ex}_{\mathcal{K}}(g, \mathbb{C}) = \mathrm{Ex}_{\mathcal{K}}(f, \mathbb{C})$. The proposition above shows that F and $f_{\mathbb{C}}$ are $\mathcal{A}$-equivalent in a neighbourhood of $t = 0$ whereas F and $g_{\mathbb{C}}$ are $\mathcal{A}$-equivalent near $t = 1$. Thus it is sufficient to show that the (open) set V of points $t_0 \in \mathbb{C}$, where F and $F_{t_0} = (1-t_0)f + t_0 g$ are $\mathcal{A}$-equivalent in a neighbourhood of t_0, is connected. We claim that the complement of this set is countable. Consider

$$\mathcal{E}x_{\mathcal{A}}(F) := \mathcal{O}^p_U / (\mathrm{Im}(J_F) + F^*(\mathfrak{m}_{\mathbb{C}^p} \mathcal{O}^p_{\mathbb{C}^p \times \mathbb{C}, 0})),$$

which is a sheaf in some neighbourhood U of $0 \times \mathbb{C}$ in $X \times \mathbb{C}$. The Kodaira-Spencer class $\delta_{KS}(\partial/\partial t)$ of F may be considered as a section in $\mathcal{E}x_{\mathcal{A}}(F)$. By assumption

$$\mathcal{E}x_{\mathcal{K}}(F) := \mathcal{O}^p_U / (\mathrm{Im}(J_F) + \sum F_i \mathcal{O}^p_U)$$

and then also $\mathcal{O}^p_U / \mathrm{Im}(J_F)$ is quasifinite over $\mathbb{C}^p \times \mathbb{C}$ at $(0,0) \in X \times \mathbb{C}$, and since $\mathrm{Ex}_{\mathcal{K}}(f, \mathbb{C}) = \mathrm{Ex}_{\mathcal{K}}(g, \mathbb{C})$, it is also quasifinite over $\mathbb{C}^p \times \mathbb{C}$ at $(0,1)$. The points where a map is quasifinite, are just the points where the fibre dimension is 0. Therefore we can apply Remmert's semicontinuity theorem [**Fis**, 3.6] to obtain that the set U' of points $(x,t) \in U$ where $\mathcal{O}^p_U / \mathrm{Im}(J_F)$ is quasifinite over $\mathbb{C}^p \times \mathbb{C}$, is Zariski open in U. Applying 4.4.10 below to

$$\begin{tikzcd}
& Z = (0 \times \mathbb{C}) \cap U' \arrow[dl] \arrow[dr] & \\
U' \arrow[rr, "F|U'"] & & Y := \mathbb{C}^p \times \mathbb{C}
\end{tikzcd}$$

and the natural map

$$u : (F|U')^{-1}(\mathcal{O}^p_Y) \to \mathcal{O}^p_{U'} / \mathrm{Im}(J_F),$$

shows that the section $\delta_{K/S}(\partial/\partial t)$ in $\mathcal{E}x_{\mathcal{A}}(F) = \mathrm{Coker}(u)$ vanishes on a nonempty set that is Zariski-open in some Zariski-open subset of Z. Thus the complement of the set $V \subseteq \mathbb{C}$ above is countable as claimed. □

LEMMA 4.4.10. *Let*

$$\begin{tikzcd}
& Z \arrow[dl, "i"'] \arrow[dr, "j"] & \\
X \arrow[rr, "F"] & & Y
\end{tikzcd}$$

be a diagram of complex spaces such that i, j *are closed embeddings. Let* $u : F^*(\mathcal{N}) \to \mathcal{M}$ *be a morphism of* $\mathcal{O}_X$*-modules, where* $\mathcal{N} \in \mathbf{Coh}(Y)$ *and* $\mathcal{M} \in \mathbf{Coh}(X)$*. Assume that*

$$\mathrm{supp}(\mathcal{M}) \cap F^{-1}(j(Z))$$

is quasifinite over Y*. Then for a section* $m \in \Gamma(X, \mathcal{M})$ *the set*

$$V = \{z \in Z : m \in \mathrm{Im}(\mathcal{N}_{j(z)} \to \mathcal{M}_{i(z)})\}$$

cuts out a Zariski-open subset in some Zariski-open dense subset W *of* Z*.*

Proof. Replacing X by $\operatorname{supp}(\mathcal{M})$ equipped with the structure given by the coherent ideal sheaf $\operatorname{Ann}(\mathcal{M}) \subseteq \mathcal{O}_X$, we may assume that $Z' := F^{-1}(j(Z))$ is quasifinite over $Z \cong j(Z)$. Obviously we may assume that Z is reduced, and we equip Z' with its reduced structure. Then the ramification locus $\Gamma \subseteq Z'$ of $F : Z' \to Z \cong j(Z)$ is analytic and nowhere dense in Z'. Let $Z'' \subseteq Z'$ be the union of all irreducible components which are not contained in $i(Z)$. Replacing X by $X \backslash (\Gamma \cup Z'')$ and Z by $Z \backslash \Gamma$ we may assume that $F : Z' \xrightarrow{\sim} j(Z)$ is an isomorphism. Let $\mathcal{O}_X|Z = i^{-1}(\mathcal{O}_X)$ be the topological inverse image of $\mathcal{O}_X$, similarly for $\mathcal{O}_Y|Z, \mathcal{M}|Z, \mathcal{N}|Z$. By 4.4.11 below the sheaves of rings $\mathcal{O}_X|Z$, $\mathcal{O}_Y|Z$ are coherent, and $\mathcal{M}|Z$, $\mathcal{N}|Z$ are coherent modules over $\mathcal{O}_X|Z$, $\mathcal{O}_Y|Z$ respectively.

We claim that $\mathcal{O}_X|Z$ is a coherent $\mathcal{O}_Y|Z$-module. This is a local problem around a point $z \in Z$, and so after replacing X, Y by suitable neighbourhoods of z in X resp. Y we are reduced to the case that F is finite. By now $F_*(\mathcal{O}_X)$ is a coherent $\mathcal{O}_Y$-module and so $F_*(\mathcal{O}_X)|Z$ is coherent over $\mathcal{O}_Y|Z$. As $Z' = F^{-1}(j(Z)) \to j(Z)$ is bijective the canonical map $F_*(\mathcal{O}_X)|Z \to \mathcal{O}_X|Z$ is an isomorphism in every stalk and so $F_*(\mathcal{O}_X)|Z \simeq \mathcal{O}_X|Z$ is coherent over $\mathcal{O}_Y|Z$, proving the claim.

Let $\bar{m} \cdot \mathcal{O}_Y|Z \subseteq (\mathcal{M}|Z)/(\mathcal{N}|Z)$ be the subsheaf generated by the class $\bar{m}$ of the given section $m \in \Gamma(X, \mathcal{M})$. As $\mathcal{M}|Z$, $\mathcal{N}|Z$ are coherent over $\mathcal{O}_Y|Z$, this subsheaf is coherent. Thus $V = Z \backslash \operatorname{supp}(\bar{m}\mathcal{O}_Y|Z)$ is Zariski-open in Z, and we are done. □

Lemma 4.4.11. *Let $f : X \to Y$ be a morphism of topological spaces and $\mathcal{O}_Y$ a sheaf of rings on Y. Then for every coherent $\mathcal{O}_Y$-module $\mathcal{N}$ the topological inverse image $f^{-1}(\mathcal{N})$ is a coherent sheaf of $f^{-1}(\mathcal{O}_Y)$-modules.*

The *proof* follows easily from the fact that f^{-1} is an exact functor. We leave the simple argument to the reader.

Remarks 4.4.12. (1) Assume that $f, g \in \mathfrak{m}_X^2 \mathcal{O}_{X,0}^p$ and that $f^{-1}(0)$ has isolated singularity at 0. Then 4.4.9 is also valid for the group $\mathcal{A}_e$. This follows with similar arguments as in 4.4.5.

(2) The results of this sections can be generalized to unfoldings. For the proof we refer the reader to [**BFl**, sect. 5].

Remark 4.4.13. Let $f : X \to Y$ be a morphism of topological spaces and $\mathcal{M}, \mathcal{N}$ sheaves on X resp. Y. If $u : f^{-1}(\mathcal{N}) \to \mathcal{M}$ is a morphism of sheaves and $m \in \Gamma(X, \mathcal{M})$ is a section then the set

$$V = \{x \in X : m \in \operatorname{Im}(\mathcal{N}_{f(x)} \to \mathcal{M}_x)\}$$

is open in X. We do not know whether this set is *Zariski*-open if X, Y are complex spaces and $\mathcal{M}, \mathcal{N}$ are coherent. In the algebraic case this is tautologically true as V is open hence Zariski-open.

Remark 4.4.14. Evtl. Problem aus der Dimca Arbeit erwaehnen.

4.5. Versal Unfoldings

In the last section of this chapter we will examine versality of unfoldings. Our main result will be that every unfolding — for each of our standard groups — admits a versal unfolding provided that its space of infinitesimal deformations over the double point is a finite dimensional vector space. We will even do this for quite general groups acting on unfoldings. However, the versal unfoldings that we construct are not versal in the full sense of our definition, see 3.4.8, but only in a restricted sense. To be precise, our main result is the following theorem.

THEOREM 4.5.1. *Let G be one of the groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{K}$, $\mathcal{C}$ or one of the associated extended groups. If $f : (X, 0) \to (\mathbb{C}^p, 0)$ is a mapping germ with $k := \dim_{\mathbb{C}} \mathrm{Ex}_G(f, \mathbb{C}) < \infty$ then f admits a deformation $F \in E(S, 0)$ over some smooth germ $(S, 0)$ of dimension k such that every other unfolding of f can be obtained from F by base change, i.e. if H is another unfolding over some germ $(T, 0)$ then there is a morphism $\alpha : (T, 0) \to (S, 0)$ such that H and $\alpha^*(F)$ are G-equivalent.*

In particular it follows that H is a formally semiuniversal deformation of f in $\mathbf{E}_G$, see 3.4.17. What is missing for versality is the full lifting property (FV) for arbitrary embeddings $(T, 0) \hookrightarrow (T', 0)$ in 3.4.8. We remark that one can also proof that there are versal deformations but all the proofs we know require Banach analytic methods which are behind the scope of this books. We refer the reader to e.g. [**BKo**], or [**Fle**].

To start with the proof we formulate first three properties on a group G acting on unfoldings that will enshure the existence of versal deformations in our weaker sense.

4.5.2. Let E : **Germs** $\to$ **Groups** be the functor of unfoldings and G a group acting on E satisfying the assumptions of 4.1.7 so that $\mathbf{E}_G \to$ **Germs** is a deformation theory. We assume that for a fixed element $f \in E(0)$ the following conditions are satisfied.

(1) For every deformation $F \in E(S, 0)$ of f the module $\mathrm{Ex}_G(F, \mathcal{O}_S)$ is finite over $\mathcal{O}_{S,0}$.
(2) $\mathbf{E}_G \to$ **Germs** admits integration of vector fields.

EXAMPLE 4.5.3. Let G one of the standard groups considered in 4.5.1. Assume that $f : (X, 0) \to (\mathbb{C}^p, 0)$ is an element in $E(0)$ such that $\mathrm{Ex}_G(f, \mathbb{C}) = \mathcal{O}^p_{X,0}/TG(f)$ is a finite dimensional $\mathbb{C}$-vector space. *Then the conditions* (1) and (2) *in* 4.5.2 *are satisfied for E and G.* In fact (2) holds by 4.2.11. In order to prove (1) observe that

$$\mathrm{Ex}_G(f, \mathcal{O}_{S,0}) = \mathcal{O}^p_{X\times S,0}/TG(f)$$

is an $\mathcal{O}_{\mathbb{C}^p\times S,0}$-module, which for each of the groups G in question can be written as M/N, where M is a finite $\mathcal{O}_{X\times S,0}$-module and N is finite over $\mathcal{O}_{\mathbb{C}^p\times S,0}$. As by assumption $M/(N + \mathfrak{m}_S M)$ is finite dimensional over $\mathbb{C}$ it follows that $M/\mathfrak{m}_S M$ is finite over $\mathcal{O}_{\mathbb{C}^p,0}$. Thus by Weierstraß' preparation theorem M is finite over $\mathcal{O}_{\mathbb{C}^p\times S,0}$. Applying again the preparation theorem to the finite $\mathcal{O}_{\mathbb{C}^p\times S,0}$-module M/N it follows that M/N is $\mathcal{O}_{S,0}$-finite.

Now 4.5.1 follows from the following more general result.

THEOREM 4.5.4. *Let $E, G, f \in E(0)$ be as in 4.5.2. Then f admits a deformation $F \in E(S, 0)$ over a smooth germ of dimension* $\dim \mathrm{Ex}_G(f, \mathbb{C})$ *such that every other deformation $H \in E(T, 0)$ can be obtained from F by base change $\alpha : (T, 0) \to (S, 0)$, i.e. $H = \alpha^*(F)$.*

The proof of 4.5.4 will be based on a lemma and proposition that we formulate and prove first.

LEMMA 4.5.5. *Let E, G and f be as in 4.5.2 and $F \in \mathbf{E}_G(S)$ a deformation of f over a germ $(S, 0)$. Then the following holds.*

(1) *The functor $\mathcal{N} \mapsto \mathrm{Ex}_G(F/S, \mathcal{N})$ on $\mathbf{Coh}(S)$ is right exact.*
(2) $\mathrm{Ex}_G(F/S, \mathcal{N}) \cong \mathrm{Ex}_G(F/S, \mathcal{O}_S) \otimes_{\mathcal{O}_S} \mathcal{N}$.

(3) *If* $S' \subseteq S$ *is a closed subspace and* $\mathcal{N}$ *a finite* $\mathcal{O}_{S'}$*-module then with* $F' := F|S'$

$$\mathrm{Ex}_G(F/S, \mathcal{N}) \cong \mathrm{Ex}_G(F'/S', \mathcal{N}).$$

PROOF. That the functor considered in (1) is halfexact is contained in 3.3.3 (3). Moreover, that it is right exact follows from the fact that $E(S[\mathcal{N}]) \to E(S[\mathcal{N}''])$ is surjective for any surjective morphism $\mathcal{N} \to \mathcal{N}''$ of $\mathcal{O}_S$-modules. (2) holds for any right exact functor on $\mathbf{Coh}(S)$, see [**Har**, III 12.5]. Finally, (3) holds generally for any deformation theory, see 3.3.6 □

PROPOSITION 4.5.6. *Let* $S = (S, 0)$ *be a smooth germ,* $H \in E(S \times \mathbb{C}^r, 0)$ *a deformation of* f *and set* $F := H|(S \times 0)$. *Assume that the Kodaira-Spencer map*

$$\delta_{KS} : \mathrm{Der}(\mathcal{O}_S, \mathbb{C}) \to \mathrm{Ex}_G(a/S, \mathbb{C})$$

is surjective. Then there is a map $\pi : (S \times \mathbb{C}^r, 0) \to (S, 0)$ *such that* H *and* $\pi^*(F)$ *are* G*-equivalent by an element of* $G(S \times \mathbb{C}^r, 0)$ *that restricts to the identity over* $S \cong S \times 0$.

PROOF. It is clearly sufficient to treat the case $r = 1$. Let $t \in \mathcal{O}_{\mathbb{C},0}$ be the coordinate function and $\{\vartheta_i\}$ a basis for $\mathrm{Der}(\mathcal{O}_S, \mathcal{O}_S)$. Then $\{\vartheta_i \otimes 1\}$ forms a basis of the $\mathcal{O}_{S\times\mathbb{C},0}$-module

$$\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{S\times\mathbb{C},0}, \mathcal{O}_{S\times\mathbb{C},0}) = \mathrm{Der}(\mathcal{O}_S, \mathcal{O}_S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S\times\mathbb{C},0} \subseteq \mathrm{Der}_(\, \mathcal{O}_{S\times\mathbb{C},0}, \mathcal{O}_{S\times\mathbb{C},0}).$$

Under the Kodaira-Spencer map

$$\delta_{KS} : \mathrm{Der}(\mathcal{O}_{S\times\mathbb{C},0}, \mathcal{O}_{S\times\mathbb{C},0}) \to \mathrm{Ex}_G(H/S \times \mathbb{C}, \mathcal{O}_{S\times\mathbb{C},0}),$$

the elements $\vartheta_i \otimes 1$ are mapped onto a generating set of $\mathrm{Ex}_G(H/S \times \mathbb{C}, \mathcal{O}_{S\times\mathbb{C},0})$, since this is true modulo $\mathfrak{m}_{S\times\mathbb{C}}$ by assumption, see 4.5.5 (2), (3). Thus there is an equation

$$\delta_{KS}(\frac{\partial}{\partial t}) = \delta_{KS}\left(\sum_i a_i \vartheta_i \otimes 1\right),$$

for some $a_i \in \mathcal{O}_{S\times\mathbb{C},0}$. Accordingly, the derivation $\delta := \partial/\partial t - \sum_i a_i(\vartheta_i \otimes 1)$ maps to zero under δ_{KS} and satisfies $\delta(t) = 1$. Integration of vector fields yields the claim. □

PROOF OF 4.5.4. . Let V be the vector space dual to $\mathrm{Ex}_G(f, \mathbb{C})$, and S_1 the trivial extension of the point 0 by V. Let $f_1 \in \mathbf{E}_G(S_1)$ be a deformation of f inducing the canonical element in $\mathrm{Ex}_G(f, V)$, i.e. the element corresponding to id_V under the canonical isomorphism

$$\mathrm{Ex}_G(f, V) \cong \mathrm{Ex}_G(f, \mathbb{C}) \otimes_{\mathbb{C}} V \cong \mathrm{Hom}_{\mathbb{C}}(V, V).$$

The element f_1 can be lifted to an element $F \in E(S, 0)$ if $(S, 0)$ is a smooth germ with first infinitesimal neighbourhood S_1. By construction the Kodaira-Spencer map

$$\delta_{KS} : \mathrm{Der}(\mathcal{O}_S, \mathbb{C}) \to \mathrm{Ex}_G(F, \mathbb{C})$$

is bijective. We claim that F satisfies the required property, i.e. every deformation $H \in E(T)$ of f is induced from F over a suitable map $(T, 0) \to (S, 0)$. If $(T, 0)$ is a closed subspace of a smoth germ then H can be lifted to that smooth germ. Therefore we may assume that $(T, 0)$ is smooth. Consider the product $S \times T$ and denote by pr_i, $i = 1, 2$ the projection onto the i-th factor and by pr the morphism to 0. Using the group stucture on E we can form $\Phi := pr_1^*(F) + pr_2^*(H) - pr^*(f)$. By construction, Φ induces F, H on $S \cong (S \times 0$ resp. $T \cong 0 \times T$. Using 4.5.6 there is a map $\pi : (S \times T, 0) \to (S, 0)$ such that Φ and $\pi^*(F)$ are G-equivalent by an element of $G(S \times T, 0)$ restricting to the identity on $S \cong S \times 0$. Thus, if $\varphi : (T, 0) \to (S, 0)$ is the composition of $T \cong 0 \times T \hookrightarrow S \times T$ and π, we get $\varphi^*(F) \cong H$ in $\mathbf{E}_G$ with an isomorphism inducing the identity on f. □

4.5.7. The proof also shows how to compute the semiuniversal unfolding in the weak sense of 4.5.1 of a map germ $f : (X, 0) \to (\mathbb{C}^p, 0)$. For this take elements $g_1, \dots, g_m \in \mathcal{O}_{X,0}^p$ which form a basis of $\mathcal{O}_{X,0}^p / TG(f)$. Then

$$F(z, s_1, \dots, s_m) := f(z) + \sum_{\mu=1}^{m} s_\mu g_\mu$$

is an unfolding as in 4.5.1.

EXAMPLE 4.5.8. Let us compute the semiuniversal unfolding (in the weak sense) of the polynomial $f(z) = z^n$ in one variable z. Here $\mathrm{Ex}_G(f, \mathbb{C}) \cong \mathcal{O}_{\mathbb{C},0}/TG(f)$. We get

$$T\mathcal{K}(f) = T\mathcal{R}(f) = T\mathcal{C}(f) = z^n \mathcal{O}_{\mathbb{C},0}$$

whereas

$$T\mathcal{K}_e(f) = T\mathcal{R}_e(f) = z^{n-1}\mathcal{O}_{\mathbb{C},0}.$$

The (formally) semiuniversal unfoldings are given by

$F(z, a_0, \dots, a_{n-1}) := z^n + a_{n-1}z^{n-1} + \dots + a_0$ in the cases $G = \mathcal{K}$, $\mathcal{R}$, $\mathcal{C}$,

and by

$F(z, a_0, \dots, a_{n-2}) := z^n + a_{n-2}z^{n-2} + \dots + a_0$ in the cases $G = \mathcal{K}_e$, $\mathcal{R}_e$.

In the cases $G = \mathcal{L}$, $\mathcal{L}_e$ the group $TG(f)$ can be identified with $\mathbb{C}\{z^n\} \subseteq \mathbb{C}\{z\}$ resp. $z\mathbb{C}\{z^n\}$. As the cokernel is not finite dimensional there is no versal deformation. In the case $G = \mathcal{A}$ the (formally) semiuniversal unfolding is the same as for $\mathcal{R}$-equivalence. Finally, for $\mathcal{A}_e$-equivalence $T\mathcal{A}_e(f) = \mathbb{C} + z^{n-1}\mathbb{C}\{z\}$. Hence in this case $F(z, a_1, \dots, a_{n-2}) := z^n + a_{n-2}z^{n-2} + \dots + a_1 z$ is the (formally) semiuniversal deformation.

Take now an arbitrary unfolding $H(z, s)$ over some base space $(S, 0)$ with $H(z, 0) \neq 0$ so that $H(z, 0)$ is $\mathcal{C}$-equivalent to z^n for some $n \in \mathbb{N}$. We obtain H by base change from the (formally) semiuniversal deformation. With other words, we find a unit $u \in \mathcal{O}_{\mathbb{C}\times S,0}$ such that H can be written in the form $H = uP$ with $P = z^n + a_{n-1}(s)z^{n-1} + \dots + a_0(s)$, whence we get back the Weierstra' preparation theorem. Similarly, H is right equivalent to a polynomial P as above, and $\mathcal{R}_e$-equivalent even to one with $a_{n-1} = 0$.

As another example we show how one can derive the classical Morse lemma in a simple way. Let A denote the ring $\mathcal{O}_{\mathbb{C}^n,0}$ and $\mathfrak{m}$ its maximal ideal.

PROPOSITION 4.5.9. (Morse lemma) *Every function $f \in \mathfrak{m}^2$ is $\mathcal{R}$-equivalent to a function of the form*

$$z_1^2 + \cdots z_k^2 + g(z_{k+1}, \dots, z_n)$$

for some k where $g(z_{k+1}, \dots, z_n) \in \mathfrak{m}^3$.

PROOF. Write $f = q + h$ where q is a quadratic polynomial and $h \in \mathfrak{m}^3$. After coordinate change — which corresponds to an $\mathcal{R}$-equivalence — we may assume that $q = z_1^2 + \dots + z_k^2$. Write $z' := (z_1, \dots, z_k)$ and $z'' := (z_{k+1}, \dots, z_n)$. The function q on $(\mathbb{C}^k, 0)$ is 2-determined by **??**. Hence q and $f(z', 0) = q + h(z', 0)$ are $\mathcal{R}$-equivalent under some right equivalence $\varphi \in \mathcal{R}(\mathbb{C}^k, 0)$. Applying $\varphi \times id_{\mathbb{C}-n-k}$ to f we are reduced to the case that $f(z', 0) = q$.

With other words, f can be considered as an unfolding of q over the space $(S, 0) := (\mathbb{C}^{n-k}, 0)$. The (formally) versal deformation is given by $Q(z_1, \dots, z_k, t) := q + t$ which is an unfolding of q over $(\mathbb{C}, 0)$. By versality, there is map $g : (S, 0) \to (\mathbb{C}, 0)$ such that f is $\mathcal{R}$-equivalent to $g^*(Q) = q + g$. □

The most important application of the preceding results is that complete intersections with isolated singularity admit a semiuniversal deformation in the weaker sense.

Theorem 4.5.10. *Let $(X_0, 0)$ be a complete intersection with an isolated singularity. Then there is a deformation $f : (X, 0) \to (S, 0)$ such that every other deformation of $(X_0, 0)$ is obtained from f by base change and which is formally semiuniversal.*

Proof. Choose an embedding $(X_0, 0) \hookrightarrow (\mathbb{C}^n, 0)$ such that $(X_0, 0)$ is given by equations $f_1, \dots, f_r \in \mathcal{O}_{\mathbb{C}^n,0}$ where $r = n - \dim X$. Then the unfoldings of $f := (f_1, \dots, f_r)$ are in 1-1 correspondence with the deformations of $(X_0, 0)$. More precisely, the category of all unfoldings of f is equivalent with the category of all deformations of $(X_0, 0)$. Using 4.5.1 the result follows. □

Example 4.5.11. Let us consider the simple example of an isolated hypersurface singularity $(X_0, 0) \subseteq (\mathbb{C}^n, 0)$ given by $f \in \mathcal{O}_{\mathbb{C}^n,0}$. To compute the semiuniversal unfolding (in the weak sense of 4.5.10) of f we choose elements, say $g_1, \dots, g_m \in \mathcal{O}_{\mathbb{C}^n,0}$, forming a basis of the vector space $\mathcal{O}_{\mathbb{C}^n,0}/\operatorname{jac}^e(f)$. By 4.5.7 the formally semiuniversal unfolding is given by

$$F(z, s_1, \dots, s_m) := f(z) + \sum_{\mu=1}^{m} s_\mu g_\mu.$$

Accordingly, the formally semiuniversal deformation of X_0 is

$$\pi : (X, 0) := \big(F^{-1}(0), 0\big) \longrightarrow (\mathbb{C}^m, 0)$$

where π is the projection.

CHAPTER 5

Properties of Versal Deformations

5.1. Smoothness of the basis of the versal deformations

In this section we will develop criteria for when the basis of the (formally) semiuniversal deformation is smooth. Our approach is based on a criterion essentially due to Z. Ran which works generally for any deformation theory. It states that the basis is smooth as soon as the Ex-functors describing infinitesimal deformations are right exact. There are many applications including deformations of Calabi-Yau manifolds, manifolds with $H^2(X,\Theta_X)=0$ and deformations of modules to which we return later. We emphasize that for this approach no obstruction theory is needed.

In the first result we consider the following setup. Let $p:\mathbf{F}\to\mathbf{An}_{(\Sigma,0)}$ be a local deformation theory and assume that $a_0\in\mathbf{F}(0)$ admits a formally semiuniversal deformation $\bar a\in\hat F(\overline{S},0)$, see **??**. As in 3.3.1 we have the functors

$$\begin{array}{ll}\mathrm{Ex}_{(\Sigma,0)}(b/T,\mathcal{M}), & \mathrm{Ex}_{(\Sigma,0)}(b,\mathcal{M})\\ \mathrm{Aut}_{(\Sigma,0)}(b/T,\mathcal{M}), & \mathrm{Aut}_{(\Sigma,0)}(b,\mathcal{M})\end{array}$$

for any $b\in\mathbf{F}$ lying over $(T,0)$.

Theorem 5.1.1. *Assume that for every $b\in\mathbf{F}(T)$ over an artinian basis $T=(T,0)\in\mathbf{An}_{(\Sigma,0)}$ the functor*

$$\mathcal{M}\longmapsto\mathrm{Ex}_{(\Sigma,0)}(b/T,\mathcal{M})$$

is right exact on $\mathbf{Coh}(T)$. Then $\bar S$ is smooth over a closed subspace of $(\hat\Sigma,0)$.

Proof. Let $\mathcal{M}$ be an artinian $\mathcal{O}_{\bar S}$–module with $\mathfrak{m}_{\bar S}^{n+1}\mathcal{M}=0$. By 3.4.14 (1b)

$$\mathrm{Ex}_{(\Sigma,0)}(\bar a/\bar S,\mathcal{M})\cong\mathrm{Ex}_{(\Sigma,0)}(a_n/S_n,\mathcal{M}),$$

where S_n denotes the n^{th} infinitesimal neighbourhood. Therefore the functor $\mathcal{M}\longmapsto\mathrm{Ex}_{(\Sigma,0)}(\bar a/\bar S,\mathcal{M})$ is right exact on the artinian $\mathcal{O}_{\bar S}$–modules. For $n\geq 1$ consider the diagram

$$\begin{tikzcd}
\mathrm{Der}_{(\Sigma,0)}\left(\mathcal{O}_{\bar S},\mathcal{O}_{S_n}\right) \arrow[r, "(\delta_{KS})_n"] \arrow[d, "\alpha_n"] & \mathrm{Ex}_{(\Sigma,0)}\left(\bar a/\bar S,\mathcal{O}_{S_n}\right) \arrow[r] \arrow[d, "\beta_n"] & \mathrm{Ex}_{(\Sigma,0)}\left(\bar a,\mathcal{O}_{S_n}\right) \arrow[d] \\
\mathrm{Der}_{(\Sigma,0)}(\mathcal{O}_{\bar S},\mathcal{O}_{S_0}) \arrow[r, "(\delta_{KS})_0"] & \mathrm{Ex}_{(\Sigma,0)}(\bar a/\bar S,\mathcal{O}_{S_0}) \arrow[r] & \mathrm{Ex}_{(\Sigma,0)}(\bar a,\mathcal{O}_{S_0}).
\end{tikzcd}$$

Since $\bar a$ is formally semiuniversal the map $(\delta_{KS})_0$ is bijective. By 3.4.15 the module $\mathrm{Ex}_{(\Sigma,0)}(\bar a,\mathcal{O}_{S_n})$ vanishes and so $(\delta_{KS})_n$ is surjective. By assumption β_n is surjective and so α_n is surjective too. Now the result follows from Lemma 5.1.2 below. ☐

Lemma 5.1.2. *Let $A\to B$ be a morphism of complete analytic $\mathbb{C}$-algebras. Then the following conditions are equivalent.*

(1) *There is an ideal* $\mathfrak{a} \subseteq A$ *and an isomorphism of* A*–algebras*

$$B \cong (A/\mathfrak{a})[[T_1, \dots T_k]] \quad \textit{for some } k \geq 0$$

(2) $\Omega^1_{B/A}$ *is a free* B*–module.*
(3) $\mathrm{Der}_A(B, B) \to \mathrm{Der}_A(B, \mathbb{C})$ *is surjective.*
(4) $\mathrm{Der}_A(B, B_n) \to \mathrm{Der}_A(B, \mathbb{C})$ *is surjective for all* n*, where* $B_n := B/\mathfrak{m}_B^{n+1}$*.*

Proof. The equivalence of (1) and (2) was shown in 2.1.3. If (2) is satisfied then

$$M \longmapsto \mathrm{Der}_A(B, M) \cong \mathrm{Hom}_B(\Omega^1_{B/A}, M)$$

is an exact functor on the category of B-modules and so (3) follows. The implication (3)⇒(4) is obvious. For the proof of (4)⇒(2) we remark first that

$$\mathrm{Hom}_B(\Omega^1_{B/A}, M) \cong \mathrm{Hom}_{B_n}(\Omega^1_{B/A} \otimes_B B_n, M)$$

The assumption therefore implies that

$$\mathrm{Hom}_{B_n}(\Omega^1_{B/A} \otimes_B B_n, B_n) \to \mathrm{Hom}_{B_n}(\Omega^1_{B/A} \otimes_B B_n, B_0)$$

is surjective. It is an easy exercise to show that then $\Omega^1_{B/A} \otimes_B B_n$ is free over B_n. Since this is true for any $n \geq 1$ it follows that $\Omega^1_{B/A}$ is free as B–module. □

Remark 5.1.3. Let T, $b \in \mathbf{F}(T)$ be as in 5.1.1. Then the following conditions are equivalent.

(1) $\mathcal{M} \mapsto \mathrm{Ex}_{(\Sigma,0)}(b/T, \mathcal{M})$ is right exact on $\mathbf{Coh}(S)$.
(2) For every surjection $\alpha : \mathcal{M} \to \mathcal{M}''$ with $\ker \alpha \cong \mathcal{O}_T/\mathfrak{m}_T$ the induced map $\mathrm{Ex}_{(\Sigma,0)}(b/T, \mathcal{M}) \to \mathrm{Ex}_{(\Sigma,0)}(b/T, \mathcal{M}'')$ is surjective.

This follows from the fact that every surjection $\mathcal{M} \to \mathcal{M}''$ can be written as a composition of maps $\mathcal{M} = \mathcal{M}_0 \to \mathcal{M}_1 \to \dots \to \mathcal{M}_n = \mathcal{M}''$, where $\mathcal{M}_{\nu-1} \to \mathcal{M}_\nu$, $1 \leq \nu \leq n$, is surjective with a kernel isomorphic to $\mathcal{O}_T/\mathfrak{m}_T$.

To show the usefulness of this remark, consider a complex manifold X satisfying $H^2(X, \Theta_X) = 0$. Let $\pi : \mathfrak{X} \to T$ be a deformation of X over an artinian germ $T = (T, 0)$. By **??**

$$\mathrm{Ex}(\mathfrak{X}/T, \mathcal{M}) \cong H^1(\mathfrak{X}, \Theta_{\mathfrak{X}} \otimes \pi^*(\mathcal{M})).$$

Let us show that this functor is right exact. By the remark above we need to verify that for every exact sequence

$$0 \to \mathcal{O}_T/\mathfrak{m}_T \to \mathcal{M} \to \mathcal{M}'' \to 0$$

the induced map β in the following induced exact cohomology sequence is surjective.

$$\dots \to H^1(\mathfrak{X}, \Theta_{\mathfrak{X}} \otimes \pi^*(\mathcal{M})) \xrightarrow{\beta} H^1(\mathfrak{X}, \Theta_{\mathfrak{X}} \otimes \pi^*(\mathcal{M}'')) \to H^2(\mathfrak{X}, \Theta_{\mathfrak{X}} \otimes \pi^*(\mathcal{O}_T/\mathfrak{m}_T)).$$

But this follows from the fact that $\Theta_{\mathfrak{X}} \otimes \pi^*(\mathcal{O}_T/\mathfrak{m}_T) \cong \Theta_X$ and our assumption. Thus we obtain the following criterion.

Corollary 5.1.4. (Kodaira-Spencer) *Let* X *be a complex manifold such that* $H^1(X, \Theta_X)$ *is a finite dimensional vector space and* $H^2(X, \Theta_X)$ *vanishes. Then* X *is unobstructed, i.e. the basis of a formally versal deformation is smooth.*

EXAMPLES 5.1.5. 1. In particular, a compact Riemann surface X of genus g has a semiuniversal deformation whose basis is smooth of dimension $\dim H^1(X, \Theta_X) = \dim H^0(X, \omega_X^{\otimes 2})$. By Riemann Roch this number equals $3g - 3$ for $g \geq 2$, equals 1 for $g = 1$ and 0 for $g = 0$.

2. Let X be a Fano manifold of dimension n, i.e. X is a compact complex manifold such that ω_X^{-1} is ample on X. Then

$$H^2(X, \Theta_X) \cong H^2\left(X, \Omega_X^{n-1} \otimes \omega_X^{-1}\right) .$$

By the Kodaira-Nakano vanishing theorem (cf. [**Wel**]) this group is zero. Thus Fano manifolds are unobstructed, i.e. the basis of a semiuniversal deformation is smooth.

3. As a special case we treat the case of del Pezzo surfaces which are the Fano manifolds of dimension 2. It is well know, see [**BPV**], that such a del Pezzo surface X is a blowing up of $\mathbb{P}^2$ in $k \leq 8$ points (including infinitesimally near points) which are generic, i.e. no 3 of them are on a line and no 5 of them on a conic. If $P_1, \ldots, P_4 \in \mathbb{P}^2$ are 4 points, and no 3 of them lie on a line, then the only automorphism of $\mathbb{P}^2$ fixing $P_1, \ldots, P_4$ is the identity. Thus, if $X = X_k$ is a blowing up of $\mathbb{P}^2$ in $k \geq 4$ points then $H^0\left(X_k, \Theta_{X_k}\right) = 0$. With a similar argument one sees that $H^0\left(X_k, \Theta_{X_k}\right) = 8 - 2k$ for $k = 0, \ldots, 4$. In order to compute $H^1\left(X_k, \Theta_{X_k}\right)$ observe that because of $H^2\left(X_k, \Theta_{X_k}\right) = 0$

$$h^1\left(X_k, \Theta_{X_k}\right) = \chi\left(\Theta_{X_k}\right) - h^0\left(\Theta_{X_k}\right) .$$

The reader may easily verify that for a blowing up $\pi : Y' \to Y$ of a smooth surface Y in a point $p \in Y$ we have $\pi_*(\Theta_{Y'}) = \mathfrak{m}_P \Theta_Y$ and $R^1\pi_*(\Theta_{Y'}) = 0$. Applying the additivity of the Euler characteristic to the exact sequence

$$0 \to \mathfrak{m}_P \Theta_Y \to \Theta_Y \to \Theta_Y / \mathfrak{m}_P \Theta_Y \cong \mathbb{C}^2 \to 0$$

gives that $\chi(\Theta_{Y'}) = \chi(\mathfrak{m}_P \Theta_Y) = \chi(\Theta_Y) - 2$. It follows that for $4 \leq k \leq 8$

$$h^1(X_k, \Theta_{X_k}) = 2(k-4),$$

whereas X_k is rigid for $0 \leq k \leq 4$.

4. Let X be a $K3$ surface which by definition is a compact complex surface with $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. Such a surface admits no global vector fields, i.e. $H^0(X, \Theta_X) = 0$ (see [**BPV**]). The pairing $\bigwedge^2 \Omega_X^1 \to \omega_X \cong \mathcal{O}_X$ provides an isomorphism $\Theta_X \cong \Omega_X^1$. Using Serre duality we get $H^2(X, \Theta_X) \cong H^0(X, \Omega_X^1)^\vee \cong H^0(X, \Theta_X)^\vee = 0$. Hence for such surfaces the basis of a versal deformation is smooth.

In many cases the Ex-groups can be identified with certain Ext-groups of coherent sheaves. To show right exactness we will often use the following technique involving the notion of grade that we now recall.

5.1.6. Let M be a finite module over a local noetherian ring A and let $\mathfrak{a}$ be an ideal in A. Then the maximal length of a M-regular sequence in $\mathfrak{a}$ is called the *grade* of M along $\mathfrak{a}$ and is denoted $\operatorname{grade}_{\mathfrak{a}} M$. We recall the following well known homological characterization of grade, see e.g. [**Mat**]: $\operatorname{grade}_{\mathfrak{a}} M \geq n$ *if and only if for all finite A-modules N with $\mathfrak{a}^k N = 0$ for $k \gg 0$ the groups*

$$\operatorname{Ext}^i_A(N, M), \quad 0 \leq i < n,$$

vanish.

Similarly, for a complex space X, a closed subspace T with ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ and a coherent module $\mathcal{M}$ on X we write

$$\operatorname{grade}_T \mathcal{M} := \min\{\operatorname{grade}_{\mathcal{J}_x} \mathcal{M}_x \mid x \in T\}.$$

Again, if $\mathcal{N}$ is a coherent $\mathcal{O}_X$-module with $\operatorname{supp} \mathcal{M} \subseteq T$ then we have

$$\mathcal{E}xt^i_X(\mathcal{N}, \mathcal{M}) = 0, \quad 0 \leq i < \operatorname{grade}_T \mathcal{M}.$$

EXAMPLE 5.1.7. If $T = \operatorname{Sing} X$, then

1. $\operatorname{grade}_T \mathcal{O}_X \geq 1$ if X is reduced,
2. $\operatorname{grade}_T \mathcal{O}_X \geq 2$ if X is normal.

For later use we note the following somewhat technical lemma.

LEMMA 5.1.8. *Let $S = (S, 0)$ be an artinian germ and $\pi : \mathfrak{X} \to S$ a holomorphic map with special fibre $X = \pi^{-1}(0)$. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_{\mathfrak{X}}$-modules with $\mathcal{F}_0 := \mathcal{F}/\mathfrak{m}_0\mathcal{F}$ and $\mathcal{G}_0 := \mathcal{G}/\mathfrak{m}_0\mathcal{G}$. Set $T_i := \operatorname{supp}(\operatorname{Tor}^{\mathfrak{X}}_i(\mathcal{F}, \mathcal{O}_X))$ and assume that*

$$\operatorname{grade}_{T_i} \mathcal{G}_0 > k - i \quad \text{for } i \geq 1. \tag{1}$$

Then the canonical map

$$\operatorname{Ext}^k_X(\mathcal{F}_0, \mathcal{G}_0) \longrightarrow \operatorname{Ext}^k_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}_0) \tag{2}$$

is surjective.

PROOF. Consider the spectral sequence

$$E_2^{pq} = \operatorname{Ext}^p_X(\mathcal{T}or^{\mathfrak{X}}_q(\mathcal{F}, \mathcal{O}_X), \mathcal{G}_0) \Longrightarrow \operatorname{Ext}^{p+q}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}_0).$$

Note that the map in (2) is the edge homomorphism $E_2^{k0} \to \operatorname{Ext}^k_{\mathfrak{X}}(\mathcal{M}, \mathcal{G}_0)$. It suffices to verify that $E_2^{pq} = 0$ for $p + q = k$, $q > 0$. But this follows from condition (1) in view of the remarks preceding this lemma. □

As an almost immediate application we get the following result.

PROPOSITION 5.1.9. *(1) Let X be a normal complex space and assume that $\operatorname{Ext}^1\left(\Omega^1_X, \mathcal{O}_X\right)$ is a finite dimensional vector space and $\operatorname{Ext}^2(\Omega^1_X, \mathcal{O}_X) = 0$. Then X is unobstructed.*

(2) If X is a complex space with $\dim H^1(X, \Theta_X) < \infty$ and $H^2(X, \Theta_X) = 0$ then the basis of a formally versal locally trivial deformation is smooth.

PROOF. For the proof of (1), let $\pi : \mathfrak{X} \to S$ be a deformation of X over an artinian base and $\mathcal{M}$ a finite $\mathcal{O}_S$–module. We need to show that the functor

$$\mathcal{M} \to \operatorname{Ext}^1_{\mathfrak{X}}\left(\Omega^1_{\mathfrak{X}/S}, \pi^*(\mathcal{M})\right)$$

is right exact. In view of the long exact Ext-sequence and 5.1.3 it is sufficient to show the vanishing of $\operatorname{Ext}^2_{\mathfrak{X}}(\Omega^1_{\mathfrak{X}/S}, \mathcal{O}_X)$; note that $\pi^*(\mathcal{O}_T/\mathfrak{m}_T) \cong \mathcal{O}_X$. For this we will apply 5.1.8 above to $\mathcal{F} = \Omega^1_{\mathfrak{X}/S}$ and $\mathcal{M} = \mathcal{O}_X$; the grade condition in loc.cit. is satisfied since X is normal and so $\operatorname{grade}_{\operatorname{Sing} X} \mathcal{O}_X \geq 2$. Hence we infer that the map

$$\operatorname{Ext}^2_X(\Omega^1_{X/S}, \mathcal{O}_X) \longrightarrow \operatorname{Ext}^2_{\mathfrak{X}}(\Omega^1_{\mathfrak{X}/S}, \mathcal{O}_X)$$

is surjective. Since the left hand side vanishes by assumption, (1) follows.

For (2) it is sufficient to show by the same argument as above, that

$$H^2\left(\mathfrak{X}, \mathcal{H}om_{\mathfrak{X}}(\Omega^1_{\mathfrak{X}/S}, \mathcal{O}_X)\right) = 0 \ .$$

This group is obviously isomorphic to $H^2(X, \Theta_X)$ and so is zero. □

Another interesting application is the following result first shown by Bogomolov, Tian and Todorov, see [**Bog**], [**Tia**], [**Tod**].

THEOREM 5.1.10. *Assume that X is a compact complex manifold which is bimeromorphically equivalent to a Kähler manifold. If $\omega_X \cong \mathcal{O}_X$ then X is unobstructed.*

Before giving the proof we remind the reader that any projective manifold is Kähler. Moreover, by [] every algebraic manifold is bimeromorphically equivalent to a projective manifold. Hence this result applies in particular to the class of algebraic manifolds.

PROOF. The proof is based on a theorem due to Deligne which we will present in an appendix of this section. Let $f : \mathfrak{X} \to S$ be a deformation of X over an artinian base and $\mathcal{M}$ a coherent $\mathcal{O}_S$–module. In a first step let us show that $\omega_{\mathfrak{X}/S} \cong \mathcal{O}_{\mathfrak{X}}$. By 5.1.16 the functor

$$\mathcal{M} \longmapsto \omega_{\mathfrak{X}/S} \otimes_{f^{-1}\mathcal{O}_S} \mathcal{M}$$

is exact on $\mathbf{Coh}\, S$. In particular the map $\omega_{\mathfrak{X}/S} \to \omega_X$ is surjective. Hence there is a nowhere vanishing section of $\omega_{\mathfrak{X}/S}$ and so $\omega_{\mathfrak{X}/S} \cong \mathcal{O}_{\mathfrak{X}}$.

By **??** there is an isomorphism

$$\mathrm{Ex}_S(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \cong H^1\left(\mathfrak{X}, \Theta_{\mathfrak{X}/S} \otimes_{\mathcal{O}_S} \mathcal{M}\right) .$$

Since $\omega_{\mathfrak{X}/S} \cong \mathcal{O}_{\mathfrak{X}}$, there are canonical isomorphisms

$$\Theta_{\mathfrak{X}/S} \cong \Omega^{n-1}_{\mathfrak{X}/S} \otimes \omega^{-1}_{\mathfrak{X}/S} \cong \Omega^{n-1}_{\mathfrak{X}/S},$$

where $n := \dim X$. By the theorem of Deligne 5.1.16 the functor

$$\mathcal{M} \longmapsto H^1\left(\mathfrak{X}, \Omega^{n-1}_{\mathfrak{X}/S} \otimes_{\mathcal{O}_S} \mathcal{M}\right) \cong H^1\left(\mathfrak{X}, \Theta_{\mathfrak{X}/S} \otimes_{\mathcal{O}_S} \mathcal{M}\right) \tag{23}$$

is exact. Hence 5.1.1 gives the result. □

EXAMPLES 5.1.11. 1. Let X be a Calabi-Yau manifold. By definition, this is a Kähler manifold with $\omega_X \cong \mathcal{O}_X$ and $H^i(X, \mathcal{O}_X) = 0$ for $i \neq 0, \dim X$. Then X is unobstructed.

2. In the same way this result applies to tori. But of course, already the well known explicit constructions yield smoothness in this case, see e.g. [**Uen**].

3. Let X be a compact complex Kähler manifold which is holomorphically symplectic, i.e. there is a holomorphic 2-form $\omega \in \Gamma(X, \Omega^2_X)$ which defines a non-degenerate skew symmetric form on the tangent space $T_{X,x}$ for every $x \in X$. Then X is unobstructed. This follows from the fact that $\omega^m \in \Gamma(X, \Omega^{2m}_X)$ generates $\omega_X = \Omega^{2m}_X$, where $n = 2m$ is the dimension of X.

Slightly more generally the following result holds.

THEOREM 5.1.12. *Let X be a compact complex manifold which is bimeromorphically equivalent to a Kähler manifold. If X admits an unramified covering $g : Z \to X$ with $\omega_Z \cong \mathcal{O}_Z$ for some $d \geq 1$, then the basis of a versal deformation of X is smooth.*

PROOF. Let $\pi : \mathfrak{X} \to S$ be a deformation of X over an artinian base $S = (S, 0)$. The topological space underlying $\mathfrak{X}$ is just X, whence the topological preimage $\mathcal{O}_{\mathfrak{Z}} := g^{-1}(\mathcal{O}_{\mathfrak{X}})$ provides a complex space $\mathfrak{Z}$ that is an unramified covering $G : \mathfrak{Z} \to$

$\mathfrak{X}$. The space Z is again bimeromorphically isomorphic to a Kähler manifold, see 5.1.13 below, and $\pi \circ G : \mathfrak{Z} \to S$ is a deformation of Z. By (23) the functor

$$\mathcal{M} \longmapsto H^1\left(\mathfrak{Z}, \Theta_{\mathfrak{Z}/S} \otimes_{\mathcal{O}_S} \mathcal{M}\right), \quad \mathcal{M} \in \mathbf{Coh}(S), \tag{24}$$

is exact. There is a natural map $\Theta_{\mathfrak{X}/S} \to G_*(\Theta_{\mathfrak{Z}/S})$ that admits a left inverse given by the trace map $\mathrm{Tr} : G_*(\Theta_{\mathfrak{Z}/S}) \to \Theta_{\mathfrak{X}/S}$ with

$$\mathrm{Tr}(\omega)(x) := \sum_{z \in \pi^{-1}(z)} \omega(z)\,.$$

Hence $\Theta_{\mathfrak{X}/S}$ is a direct summand of $G_*(\Theta_{\mathfrak{Z}/S})$ and so for $\mathcal{M} \in \mathbf{Coh}(S)$

$$H^1\left(\mathfrak{X}, \Theta_{\mathfrak{X}/S} \otimes_{\mathcal{O}_S} \mathcal{M}\right) \quad \text{is a direct summand of} \quad H^1\left(\mathfrak{Z}, \Theta_{\mathfrak{Z}/S} \otimes_{\mathcal{O}_S} \mathcal{M}\right)$$

Using (24) the functor

$$\mathcal{M} \longmapsto H^1\left(\mathfrak{X}, \Theta_{\mathfrak{X}/S} \otimes_{\mathcal{O}_S} \mathcal{M}\right), \quad \mathcal{M} \in \mathbf{Coh}(S),$$

is exact. Hence 5.1.1 gives the result. □

In the proof above we have used the following observation. For the proof we refer the reader to [**?**]. Note that this result is obvious if one restricts to algebraic manifolds, since an unramified covering of an algebraic manifold is again algebraic.

Lemma 5.1.13. *Let X be a compact complex manifold which is bimeromorphically equivalent to a Kähler manifold. If $Z \to X$ is a finite unramified covering then Z is also bimeromorphically equivalent to a Kähler manifold.*

As an application we obtain the following result, see [**?**].

Corollary 5.1.14. *Let X be a compact complex manifold which is bimeromorphically equivalent to a Kähler manifold. If $\omega_X^{\otimes d} \cong \mathcal{O}_X$ for some $d \geq 1$ then the basis of a versal deformation of X is smooth.*

Proof. The isomorphism $\omega_X^{\otimes d} \cong \mathcal{O}_X$ defines an algebra structure on $\mathcal{A} := \bigoplus_{i=0}^{d-1} \omega_X^{\otimes i}$. Let $g : Z \to X$ be the associated unramified covering so that $g_*(\mathcal{O}_Z) \cong \mathcal{A}$. The dualizing module on Z is given by $\omega_Z = g^*(\omega_X)$. As

$$g_* g^*(\omega_X) \cong \omega_X \otimes \mathcal{A} \cong \mathcal{A}$$

it follows that $\omega_Z \cong \mathcal{O}_Z$. Hence the result follows from 5.1.12. □

Examples 5.1.15. 1. Let X be a compact complex surface of Kodaira dimension 0. Then $\omega_X^{\otimes 12} \cong \mathcal{O}_X$, see [**BPV**]. Hence such a surface is unobstructed.

2. Let Z be a compact complex manifold with $\omega_Z \cong \mathcal{O}_Z$. Assume that G is a finite group of order, say, d acting on Z without fixed points, so that $X := Z/G$ is a compact complex manifold. It follows that X is unobstructed.

For a more concrete example, take the Fermat quintic Z in $\mathbb{P}^4$ given by the equation $x_0^5 + \ldots + x_4^5 = 0$. Then $\mathbb{Z}_5$ acts on Z via

$$[x_0 : \ldots : x_4] \longmapsto [\zeta^0 x_0 : \zeta^1 x_1 : \ldots : \zeta^4 x_4],$$

where $\zeta \in \mathbb{Z}_5$ is a 5^{th} root of unity. The reader may easily verify that this action has no fixed points. Since $\omega_Z \cong \mathcal{O}_Z$ the quotient $X := Z/\mathbb{Z}_5$ is unobstructed.

Appendix: Direct images of differential forms. In this appendix we will provide a proof of a result of Deligne which was used in the proof of 5.1.10 and 5.1.12. In the following $S = (S, 0)$ will denote an artinian complex space germ and $\pi : \mathfrak{X} \to S$ will be a smooth proper morphism, so that the special fibre $X = \pi^{-1}(0)$ is a compact complex manifold. Since the argument below is based on Hodge theory we must assume that X is bimeromorphic to a Kähler manifold. It is well known that for such manifolds there is a Hodge decomposition

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{pq} \quad \text{with } H^{pq} := H^q(X, \Omega_X^p),$$

see [?] for instance. The next result essentially provides a relative version of this decomposition.

Theorem 5.1.16 (Deligne). *1. For every coherent $\mathcal{O}_S$–module $\mathcal{M}$ the de Rham complex $\Omega^\bullet_{\mathfrak{X}/S} \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1}\mathcal{M}$ is a resolution of $\pi^{-1}\mathcal{M}$.*

2. $R^q\pi_*\left(\Omega^p_{\mathfrak{X}/S} \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1}\mathcal{M}\right) \cong R^q\pi_*\left(\Omega^p_{\mathfrak{X}/S}\right) \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1}\mathcal{M}$ and $R^q\pi_*\left(\Omega^p_{\mathfrak{X}/S}\right)$ is a free $\mathcal{O}_S$-module.

In particular, the functors $\mathcal{M} \mapsto R^q\pi_*\left(\Omega^p_{\mathfrak{X}/S} \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1}\mathcal{M}\right)$ are exact.

Proof. (1) is seen by an easy induction on the length of $\mathcal{M}$ using the fact that an exact sequence of $\mathcal{O}_S$–modules $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ induces an exact sequence of complexes

$$0 \to \Omega^\bullet_{\mathfrak{X}/S} \otimes_{\mathcal{O}_S} \mathcal{M}' \to \Omega^\bullet_{\mathfrak{X}/S} \otimes \mathcal{M} \to \Omega^\bullet_{\mathfrak{X}/S} \otimes \mathcal{M}'' \to 0 \ .$$

For the proof of (2) observe that there is a spectral sequence

$$(*) \qquad E_2^{pq} = H^q\left(\mathfrak{X}, \Omega^p_{\mathfrak{X}/S} \otimes_{\mathcal{O}_S} \mathcal{M}\right) \Longrightarrow H^{p+q}\left(\mathfrak{X}, \pi^{-1}\mathcal{M}\right) \ .$$

The sheaf $\pi^{-1}(\mathcal{M})$ on $\mathfrak{X}$ is the constant sheaf associated to the $\mathcal{O}_{S,0}$ module $\mathcal{M}$, and $\mathfrak{X}$ and X have the same underlying topological spaces. By the universal coefficient theorem

$$H^k\left(\mathfrak{X}, \pi^{-1}(\mathcal{M})\right) \cong H^k(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{M}.$$

It follows that the length of the right hand side of $(*)$ is given by the product $\dim_{\mathbb{C}}(\mathcal{M}) \dim_{\mathbb{C}} H^k(X, \mathbb{C})$ and so by Hodge theory

$$(**) \qquad \lg H^k\left(\mathfrak{X}, \pi^{-1}(\mathcal{M})\right) = \lg \mathcal{M} \cdot \sum_{p+q=k} \dim_{\mathbb{C}} H^q\left(X, \Omega^p_X\right) \ .$$

Let us now compute the length of the right hand side of $(*)$. Using 5.1.17 below we get

$$(***) \qquad \lg H^q\left(\mathfrak{X}, \Omega^p_{\mathfrak{X}/S} \otimes_{\mathcal{O}_S} \mathcal{M}\right) \le \lg \mathcal{M} \cdot \dim_{\mathbb{C}} H^q\left(X, \Omega^p_X\right).$$

On the other hand, $H^k\left(\mathfrak{X}, \pi^{-1}(\mathcal{M})\right)$ is a subquotient of $\bigoplus_{p+q=k} E_2^{pq}$. It follows that

$$\begin{array}{rcll} \lg \mathcal{M} \sum_{p+q=k} \dim_{\mathbb{C}} H^q\left(X, \Omega^p_X\right) & = & \lg H^k\left(\mathfrak{X}, \pi^{-1}(\mathcal{M})\right) & \text{by } (**) \\ & \le & \sum\limits_{p+q=k} \lg E_2^{pq} & \\ & \le & \lg \mathcal{M} \sum_{p+q=k} \dim_{\mathbb{C}} H^q\left(X, \Omega^p_X\right) & \text{by } (***) \end{array}$$

Hence all inequalities are equalities, and in particular $(***)$ is an equality for every $\mathcal{M}$. The claim follows now from 5.1.17 (2) and (3) below. $\square$

It remains to show the following lemma.

Lemma 5.1.17. *Let A be a local ring and T a half-exact functor from the category of artinian A-modules into itself. Then the following hold.*

1. *For every artinian A–module M we have $\lg T(M) \leq \lg M \lg T\,(A/\mathfrak{m}_A)$.*
2. *If in (1) equality holds for every such M then T is exact.*
3. *If A is artinian and T is exact then for every finite A-module the canonical map $T(A) \otimes M \to T(M)$ is bijective and $T(A)$ is a free A–module.*

Proof. (1) is easily seen by induction on $\lg M$ using the fact that an exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ induces a sequence $T(M_0) \to T(M_1) \to T(M_2)$ exact in the middle. For the proof of (2), assume that $\lg T(M_i) = \lg M_i \lg T(A/\mathfrak{m}_A)$ for $i = 0, 1, 2$. Then

$$\lg T(M_1) = \lg T(M_0) + \lg T(M_2)$$

and so

$$0 \to T(M_0) \to T(M_1) \to T(M_2) \to 0$$

has to be exact. Finally, to prove (3) take a presentation $A^m \to A^n \to M \to 0$. Since T is exact the top row in the diagram

$$\begin{tikzcd}
T(A^n) \arrow[r] & T(A^m) \arrow[r] & T(M) \arrow[r] & 0 \\
T(A) \otimes A^n \arrow[r] \arrow[u, "\alpha"] & T(A) \otimes A^m \arrow[r] \arrow[u, "\beta"] & T(A) \otimes M \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

is exact. As T commutes with finite direct sums the maps α, β are isomorphisms. Henceγ has to be an isomorphism too. In particular, $M \to T(A) \otimes M$ is exact and so $T(A)$ is free. $\square$

5.2. Embedded deformations

In this section we will study embedded deformations. This will be used to compare all deformations of a complex space with the embedded ones. In particular this will allow us to treat various examples. We will study the case of complete intersections in projective space more closely and are, for instance, able to determine all smooth complete intersections for which every deformation is again projective.

It is useful to introduce the following terminology.

Definition 5.2.1. Let Z be a fixed complex space. By a *family of compact subspaces* parameterized by a complex space S we mean a complex subspace $X \subseteq Z \times S$ such that the projection

$$p_2 : X \to S$$

is proper and flat.

If $T \to S$ is a morphism then by base change we get a family of compact subspaces parameterized by T

$$X \times_S T \to T\ .$$

Thus, assigning to S the set

$$\mathrm{Hilb}_Z(S) := \{X \subseteq Z \times S \mid X \text{ is a family of compact subspaces of } Z\}\ ,$$

we get a set valued functor $\mathrm{Hilb}_Z : \mathbf{An}^{op} \to \mathbf{Sets}$. This functor is called the *Hilbert moduli functor.*

The basic result about the Hilbert moduli functor is that it is representable, namely:

THEOREM 5.2.2 (Douady). Hilb_Z *is representable by a complex space* $\mathcal{H}ilb_Z = H_Z$.

In the following, we will call H_Z the *Douady space* of Z.

The reader can find a *proof* in the excellent written paper of Douady [**Dou**] or in [**BKo**], where even much more generally all the known existence theorems for versal deformations in complex analysis are proven. Another good source in the algebraic case are the Bourbaki talks of Grothendieck, see [**Gro**], and [**Vie**].

REMARK 5.2.3. Alternatively, we can also form the fibration in groupoids $p : \mathbf{Hilb}_Z \to \mathbf{An}$ given by the families of compact subspaces of Z, cf. 3.1.2. An object a of $\mathbf{Hilb}_Z$ over S consists in a family of subspaces $X \hookrightarrow Z \times S$, and a morphism into another object $b = (Y \hookrightarrow Z \times T)$ is a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "\tilde{f}"] \arrow[d, hook] & X \arrow[d, hook] \\
Z \times T \arrow[r, "id\times f"] & Z \times S,
\end{tikzcd}$$

where $f : T \to S$ is a holomorphic map. In particular, a morphism $\tilde{f} : b \to a$ is uniquely determined by the underlying morphism $f := p(\tilde{f}) : p(b) \to p(c)$. Therefore, in this case the fibration in groupoids associated to embedded deformations is equivalent to the underlying Hilbert moduli functor.

We note that this fibration in groupoids constitutes a deformation theory as follows from Schuster's result 2.4.5.

In the following we will describe an homological terms the set of infinitesimal extensions of subspaces. Let S be a fixed complex space, $\mathcal{M} \in \mathbf{Coh}(S)$ and $a = (X \hookrightarrow Z \times S)$ a family of compact subspaces of Z. Specializing the constructions of 3.3.1 to the case of the Hilbert moduli functor Hilb_Z (or, equivalently, to the associated fibration in groupoids $\mathbf{Hilb}_Z \to \mathbf{An}$) we get groups

$$\begin{array}{ccc} \mathrm{Aut}(a/S, \mathcal{M}) & , & \mathrm{Aut}(a, \mathcal{M}) \\ \mathrm{Ex}(a/S, \mathcal{M}) & , & \mathrm{Ex}(a, \mathcal{M}). \end{array}$$

As morphisms in $\mathbf{Hilb}_Z$ are uniquely determined by the underlying maps in $\mathbf{An}$ we have that $\mathrm{Aut}(a/S, \mathcal{M}) = 0$. Moreover, $\mathrm{Ex}(a/S, \mathcal{M})$ consists of all extensions X' of X by $\mathcal{M}_X := \mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{O}_X$ that fit into a diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, hook] & X' \arrow[d, hook] \\
Z \times S \arrow[r] & Z \times S[\mathcal{M}].
\end{tikzcd}$$

In other words,

$$\mathrm{Ex}(a/S, \mathcal{M}) \cong \mathrm{Ex}_{Z\times S}(X, \mathcal{M}_X).$$

Using 2.5.2 the right hand side is isomorphic to

$$\mathrm{Hom}_{Z\times S}\left(\mathcal{J}/\mathcal{J}^2, \mathcal{M}_X\right),$$

where $\mathcal{J} \subseteq \mathcal{O}_{Z\times S}$ is the ideal sheaf of X in $Z \times S$. We have shown the following proposition.

PROPOSITION 5.2.4. *For a family of compact subspaces $a = (X \hookrightarrow Z \times S)$ over S and a coherent $\mathcal{O}_S$-module $\mathcal{M}$ we have*

$$\mathrm{Aut}(a/S, \mathcal{M}) = 0 \quad \textit{and} \quad \mathrm{Ex}(a/S, \mathcal{M}) \cong \mathrm{Hom}_{Z\times S}(\mathcal{J}/\mathcal{J}^2, \mathcal{M}_X).$$

In the next result we give a first criterion for the smoothness of the Douady space. The main tool is the general criterion given in 5.1.1.

PROPOSITION 5.2.5. *Let $X \subseteq Z$ be a compact subspace with ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_Z$ and $p = [X] \in H_Z$ the associated point in the Douady space. Assume that the following conditions are satisfied.*

1. $\mathrm{Ext}^1_X(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X) = 0$.
2. $\mathrm{grade}_T\, \mathcal{O}_X \geq 1$, *where T denotes the analytic set of points where $\mathcal{J}$ is not locally generated by a regular sequence.*

Then p is a smooth point of H_Z, and $\dim_p H_Z = \dim_{\mathbb{C}} \mathrm{Hom}_X(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X)$.

PROOF. Let $\mathfrak{X} \subseteq Z\times S$ be an embedded deformation of X over an artinian base $S = (S, 0)$ with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{Z\times S}$. Note that then $\mathcal{I}$ is also locally generated by a regular sequence in the points of $X\backslash T$ (see [?]). Hence $\mathcal{I}/\mathcal{I}^2$ is locally free on $X\backslash T$ and so

$$(*) \qquad \mathrm{supp}\, \mathcal{T}or_1^{\mathfrak{X}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) \subseteq T.$$

We need to show that the functor

$$\mathbf{Coh}(T) \ni \mathcal{M} \longmapsto \mathrm{Hom}_{\mathfrak{X}}(\mathcal{I}/\mathcal{I}^2, \mathcal{M}_{\mathfrak{X}})$$

is right exact. In view of the remark 5.1.3 and the long Ext-sequence it is sufficient to verify that $\mathrm{Ext}^1_{\mathfrak{X}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = 0$. For this we will apply 5.1.8 to $k = 1$, $\mathcal{F} = \mathcal{I}/\mathcal{I}^2$ and $\mathcal{G} = \mathcal{O}_X$. The condition on the grade in loc.cit. is satisfied because of our assumption (2) and $(*)$. Hence we infer that the map

$$\mathrm{Ext}^1_X(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^1_{\mathfrak{X}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$$

is surjective. Since the left hand side vanishes, this proves the result. □

EXERCISE 5.2.6. Show that $p \in H_Z$ is a smooth point if $\mathrm{Ext}^1_Z(\mathcal{J}, \mathcal{O}_X) = 0$.

EXAMPLES 5.2.7. 1. Let $X \subseteq Z$ be locally a complete intersection with ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_Z$. Then $\mathcal{J}/\mathcal{J}^2$ is locally free over X and $\mathcal{N} := \mathcal{H}om(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X)$ is the normal bundle. By 5.2.5, if $H^1(X, \mathcal{N}) = 0$ then $[X] \in H_Z$ is a smooth point, and the dimension of H_Z at $[X]$ is equal to $h^0(X, \mathcal{N})$.

2. Let $X \subseteq Z := \mathbb{P}^n$ be a complete intersection given by homogeneous equations $f_1, \ldots, f_r$, so that $\dim X = n - r$. If $\mathcal{J} \in \mathcal{O}_{\mathbb{P}^n}$ is the ideal sheaf of X then

$$\mathcal{J}/\mathcal{J}^2 \cong \mathcal{O}_X(-d_1) \oplus \cdots \oplus \mathcal{O}_X(-d_r)$$

with $d_\varrho := \deg f_\varrho$. Hence the normal bundle $\mathcal{N}$ is isomorphic to $\bigoplus_\varrho \mathcal{O}_X(d_\varrho)$. In particular, if $\dim X \geq 2$ then $H^1(X, \mathcal{N}) = 0$ and so $[X]$ is a smooth point of $H_{\mathbb{P}^n}$ of dimension $\sum_\varrho h^0(X, \mathcal{O}_X(d_\varrho))$. (In 5.2.9 we will see that the last statement also holds if $\dim X \leq 1$.)

3. Let X be a compact Riemann surface of genus g and $D \in \mathrm{Div}(X)$ a divisor of degree $d > 2g$. Consider the embedding $X \hookrightarrow \mathbb{P} := \mathbb{P}^{d-g+1}$ given by the complete linear system $|D|$ which is given by $[s_0 : \ldots : s_{d-g}]$, where $s_0, \ldots, s_{d-g}$ are a basis of $H^0(X, \mathcal{O}_X(D))$ (see [**?**]). Observe that $H^1(X, \mathcal{O}_X(D)) = 0$. There is an exact sequence

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathbb{P}} \otimes \mathcal{O}_X \longrightarrow \mathcal{N} \longrightarrow 0 \tag{1}$$

where $\mathcal{N}$ is the normal bundle of X in $\mathbb{P}$. Using the Euler sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{d-g+1} \longrightarrow \Theta_{\mathbb{P}} \otimes \mathcal{O}_X \longrightarrow 0, \tag{2}$$

it follows that $H^1(X, \Theta_{\mathbb{P}} \otimes \mathcal{O}_X)$ is a quotient of $H^1(X, \mathcal{O}_X(1))^{d-g+1}$ and so vanishes. Therefore by the exact sequence (1) $H^1(X, \mathcal{N})$ vanishes too. Thus it follows that $[X]$ is a smooth point of $H_{\mathbb{P}}$. In order to determine the dimension we have to compute $h^0(X, \mathcal{N})$ which is given by

$$\begin{aligned} \chi(\mathcal{N}) &= \chi(\Theta_{\mathbb{P}} \otimes \mathcal{O}_X) - \chi(\Theta_X) \\ &= (d-g+1)\chi(\mathcal{O}_X(D)) - \chi(\Theta_X) - \chi(\mathcal{O}_X) \\ &= \Big((d-g+1)^2 - 1\Big) + (3g-3) + g. \end{aligned}$$

Observe that $(d-g+1)^2 - 1$ is the dimension of $\mathrm{PGL}_{d-g+1}(\mathbb{C})$ which acts on $\mathbb{P}$ and therefore also on $H_{\mathbb{P}}$. Moreover, if $g \geq 2$, the number of moduli of X is just $3g-3$, and the summand g reflects the fact that the divisor D is varying in the Jacobian of X which has dimension g.

5.2.8. Let us return to the general situation as considered in 5.2.4. In a next step we will give a more explicit description of the Kodaira Spencer map. For this consider the maps

$$\mathrm{Der}\,(\mathcal{O}_S, \mathcal{M}) \longrightarrow \mathrm{Der}\,(\mathcal{O}_{Z \times S}, \mathcal{M}_X)$$

which lifts a vector field to the product, the dual of the Jacobi map

$$\mathbf{j}^\vee : \mathrm{Der}(\mathcal{O}_{Z \times S}, \mathcal{M}_X) \longrightarrow \mathrm{Hom}_X(\mathcal{J}/\mathcal{J}^2, \mathcal{M}_X),$$

see 2.5.5, and the Kodaira Spencer map

$$\delta_{KS} : \mathrm{Der}(\mathcal{O}_S, \mathcal{M}) \longrightarrow \mathrm{Ex}(a/S, \mathcal{M}).$$

Using 3.3.10 we get a commutative diagram

$$\begin{tikzcd} \mathrm{Der}\,(\mathcal{O}_S, \mathcal{M}) \arrow[r, "\delta_{KS}"] \arrow[d] & \mathrm{Ex}(a/S, \mathcal{M}) \\ \mathrm{Der}\,(\mathcal{O}_{Z \times S}, \mathcal{M}_X) \arrow[r, "\mathbf{j}^\vee"] & \mathrm{Hom}_X(\mathcal{J}/\mathcal{J}^2, \mathcal{M}_X). \arrow[u, "\cong"'] \end{tikzcd}$$

Thus the Kodaira-Spencer map can be identified with the composition

$$\mathrm{Der}\,(\mathcal{O}_S, \mathcal{M}) \longrightarrow \mathrm{Der}\,(\mathcal{O}_{Z \times S}, \mathcal{M}_X) \xrightarrow{\mathbf{j}^\vee} \mathrm{Hom}_X(\mathcal{J}/\mathcal{J}^2, \mathcal{M}_X). \tag{$*$}$$

To be more explicit, assume that S is a closed subspace of some open subset U of $\mathbb{C}^N$ and that $\vartheta = \Sigma m_i \partial/\partial x_i$ is in $\mathrm{Der}\,(\mathcal{O}_S, \mathcal{M}) \subseteq \mathrm{Der}\,(\mathcal{O}_U, \mathcal{M})$. Assume further that $X \hookrightarrow \mathbb{P}^N \times S$ is given by polynomials $F_1, \ldots, F_r$ in $\Gamma(S, \mathcal{O}_S)\,[Z_0, \ldots, Z_n]$ which are homogeneous in $Z_0, \ldots, Z_n$. Then

$$\mathcal{J}/\mathcal{J}^2 \cong (I/I^2)^\sim,$$

where $I = (F_1, \ldots, F_r)$. Under the identifications in $(*)$, $\delta_{KS}(\vartheta)$ is the map in $\mathrm{Hom}_X(\mathcal{J}/\mathcal{J}^2, \mathcal{M}_X)$ with

$$F_\varrho \longmapsto \vartheta(F_\varrho) = \sum_i m_i \frac{\partial F(z,x)}{\partial x_i} \bmod \mathcal{J} \ .$$

Example 5.2.9. Let $X \subseteq \mathbb{P}^n$, $f_1, \ldots, f_r$ be as in 5.2.6 (2) and suppose that $\dim X \geq 1$. We choose r–tuples of homogeneous polynomials

$$g_\nu = (g_{\nu 1}, \ldots, g_{\nu r}) \in \bigoplus_{\varrho=1}^{r} H^0\left(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d_\varrho)\right), \quad i \leq \nu \leq N,$$

i.e. $g_{\nu\varrho}$ is a homogeneous polynomial of degree d_ϱ. We assume that the residue classes of $g_1, \ldots, g_N$ form a basis of the vector space $\bigoplus_\varrho H^0\left(X, \mathcal{O}_X(d_\varrho)\right)$. Consider the polynomials

$$F_\varrho(z,x) := f_\varrho(z) + \sum_{\nu=1}^{N} x_\nu g_{\nu\varrho}(z), \quad \varrho = 1, \ldots, r,$$

where $x = (x_1, \ldots, x_N) \in \mathbb{C}^N$. They are homogeneous in $Z_0, \ldots, Z_n$ of degree d_ϱ and define a subspace $\mathfrak{X} \subseteq \mathbb{P}^n \times S$ with $S := \mathbb{C}^N$. The defining equations F_ϱ, together with $x_1, \ldots, x_N$ form a regular sequence in $\mathcal{O}_{S,0}\left[Z_0, \ldots, Z_n\right]$. In particular, the F_ϱ itself form a regular sequence and so $\mathfrak{X}$ is flat over S near 0. The Kodaira–Spencer map for this family

$$\mathrm{Der}\left(\mathcal{O}_{S,0}, \mathbb{C}\right) \to \mathrm{Hom}\left(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X\right) \cong \bigoplus_{\varrho=1}^{r} H^0\left(X, \mathcal{O}_X(d_\varrho)\right)$$

is given by

$$\frac{\partial}{\partial z_\nu} \longmapsto g_\nu \bmod \mathcal{J}.$$

By construction this map is bijective. Using 3.4.17 it follows that $\mathfrak{X} \to S$ is the formally semiuniversal family at $0 \in S$. In particular we deduce that *a complete intersection X in $\mathbb{P}^n$ always defines a smooth point $[X]$ of the Douady space $H_{\mathbb{P}^n}$.*

This example generalizes. Let Z be a compact complex space and $X \subseteq Z$ locally a complete intersection of codimension r which is given as the zero set of a section σ in a vector bundle $\mathcal{E}$ of rank r over Z. Consider over $S := H^0(Z, \mathcal{E})$ the family of subspaces

$$\mathfrak{X} := \{(z,\tau) \in Z \times S \mid \tau(z) = 0\} \subseteq Z \times S,$$

which is flat and proper over S in a neighbourhood of $\sigma \in S$. Then we have the following result.

Proposition 5.2.10. *Let $\mathcal{J} \subseteq \mathcal{O}_Z$ be the ideal sheaf of X and assume that $H^1(Z, \mathcal{E} \otimes \mathcal{J}) = 0$. Then the following hold.*

1. *$\mathfrak{X} \subseteq Z \times S$ is a versal embedded deformation of $X \subseteq Z$.*
2. *The Douady space H_Z is smooth at $[X]$.*

Proof. (2) is a consequence of (1) and 3.5.6. In order to prove (1), note that S is smooth and so, in view of 3.4.17, it suffices to show that the Kodaira-Spencer map is surjective. This map can be identified with the map

$$\delta_{KS} : T_\sigma S \cong H^0(Z, \mathcal{E}) \longrightarrow \mathrm{Hom}_X(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X),$$

see 5.2.8. Notice that $\mathcal{J}/\mathcal{J}^2 \cong \mathcal{E}^\vee \otimes \mathcal{O}_X$ and so $\mathrm{Hom}_X(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X) \cong H^0(X, \mathcal{E} \otimes \mathcal{O}_X)$. Using the description of δ_{KS} given in 5.2.8 above, the reader may easily verify that this map is just given by the restriction map $H^0(Z, \mathcal{E}) \to H^0(X, \mathcal{E} \otimes \mathcal{O}_X)$. As $H^1(Z, \mathcal{E} \otimes \mathcal{J})$ vanishes this restriction map is surjective, and (1) follows. $\square$

In particular this implies that every embedded deformation of $X \hookrightarrow Z$ over a base space $(T, 0)$ is given by the set of zeros of a section $\tau \in H^=(Z \times T, p_1^*(\mathcal{E}))$ which induces σ on $Z \cong Z \times \{0\}$.

We now turn to the question as to when every (abstract) deformation of a compact subspace $X \subseteq Z$ is embeddable into Z. The standard criterion is as follows.

Proposition 5.2.11. *Assume that* $\mathrm{Ex}(Z, \mathcal{O}_X) = 0$. *Let* $\mathfrak{X} \subseteq Z \times H_Z$ *be the universal family over the Douady space* $H := H_Z$. *Then* $\mathfrak{X} \to H$ *is a versal deformation of the compact complex space* X *at* $p := [X] \in H$.

In particular it follows, that for every deformation $\mathfrak{X}' \to S'$ over some germ $(S', 0)$ there is an S-embedding $\mathfrak{X}' \subseteq Z \times S'$ near the special fibre. We will say in this case that *the deformations of* X *are embeddable* into Z.

Proof. Using the versality criterion 3.4.15 we need to show that $\mathrm{Ex}(a, \mathbb{C}) = 0$, where $a := (\mathfrak{X} \to H)$ is the (abstract) deformation of X as above and $\mathbb{C} = \mathbb{C}_p$ denotes the sheaf $\mathbb{C}$ on H concentrated in p. More concretely, we need to show that for every extension $(H, p) \hookrightarrow (H', p)$ of H by $\mathbb{C}$ and every deformation $p' : \mathfrak{X}' \to H'$ of X over H' with $\mathfrak{X}'|H \cong \mathfrak{X}$ (near p) there is a cartesian diagram

$$\begin{tikzcd}
\mathfrak{X}' \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\
H' \arrow[r, "\varrho"] & H,
\end{tikzcd}$$

so that τ and ϱ induce the identity on $\mathfrak{X}$, H respectively. Using the universal property of H it is sufficient to construct an H'-embedding $\alpha' : \mathfrak{X}' \hookrightarrow Z \times H'$ lifting the given embedding $\alpha = (\beta, p) : \mathfrak{X} \hookrightarrow Z \times H$. For this consider the fibred sum Z' in the diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, hook] \arrow[d, "\beta"'] & \mathfrak{X}' \arrow[d, "\beta'"'] \\
Z \arrow[r, hook] & Z',
\end{tikzcd}$$

so that $Z \hookrightarrow Z'$ is an extension of Z by $\mathcal{O}_X$ (cf. 2.4.5). By assumption, this extension splits, i.e. there is a retraction $r : Z' \to Z$. Then $\alpha' := (r \circ \beta', p')$ is a morphism as desired. $\square$

Let us apply this to the case of subspaces $X \hookrightarrow Z = \mathbb{P}^n$. We will show the following result.

Proposition 5.2.12. *Assume that* $X \subseteq \mathbb{P}^n$ *is arithmetically Gorenstein of dimension* ≥ 2. *Then the deformations of* X *can be embedded into* $\mathbb{P}^n$ *except when* $\dim X = 2$ *and* $\omega_X \cong \mathcal{O}_X$.

Observe that if X is smooth then in the exceptional case X is a $K3$-surface. It is well known that in this case there are always deformations of X that are not embeddable into $\mathbb{P}^n$, see 2.5.13. Alternatively, this follows from the fact that every $K3$-surface admits arbitrary small deformations into non-algebraic surfaces (see [**BPV**]).

By 2.3.6, $\mathrm{Ex}(\mathbb{P}^n, \mathcal{O}_X) \cong H^1(X, \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_X)$. Therefore the proof follows from 5.2.11 and the following vanishing lemma.

Lemma 5.2.13. *Let X be as in 5.2.12. Then $H^1(X, \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_X) = 0$.*

Proof. Consider the Euler sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \otimes V \to \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_X \to 0,$$

where $\mathbb{P}^n = \mathbb{P}(V)$, i.e. $V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))^\vee$. As $H^1(X, \mathcal{O}_X(1)) = 0$ the module $H^1(X, \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_X)$ is the kernel of the map

$$\alpha : H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X(1)) \otimes V.$$

Thus it vanishes if $\dim X \geq 3$. Now assume that $\dim X = 2$. Then α is dual to the multiplication map

$$\alpha^* : H^0(X, \omega_X(-1)) \otimes V^\vee \longrightarrow H^0(X, \omega_X).$$

Since by assumption $\omega_X \cong \mathcal{O}_X(k)$ for some k and X is in particular arithmetically normal, the map α^* is surjective unless $k = 0$. ☐

Applying the criterion above to the case of complete intersections we obtain the following result (cf. [**Ser**]).

Corollary 5.2.14. *Let $X \subseteq \mathbb{P}^n$ be a complete intersection of codimension r given by equations $f_1, \ldots, f_r$ of degree $d_1, \ldots, d_r$, where $2 \leq d_1 \leq \ldots \leq d_r$. Assume that $\dim X \geq 2$. Then every deformation of X can be embedded into $\mathbb{P}^n$ except in the following three cases.*

1. *$r = 1$, $n = 3$, $d := d_1 = 4$, i.e. X is a quartic in $\mathbb{P}^3$.*
2. *$r = 2$, $n = 4$, $(d_1, d_2) = (2, 3)$, i.e. X is an intersection of a quartic and a cubic hypersurface.*
3. *$r = 3$, $n = 5$, $(d_1, d_2, d_3) = (2, 2, 2)$, i.e. X is the intersection of three quadrics in $\mathbb{P}^5$.*

The *proof* follows immediately from 5.2.12 since X is arithmetically Gorenstein with $\omega_X \cong \mathcal{O}_X(d_1 + \ldots + d_r - n - 1)$.

Remark 5.2.15. Slightly more generally as in 5.2.12 the following holds. Assume that $X \subseteq \mathbb{P}^n$ is a subscheme such that the following conditions are satisfied.

1. $H^1(X, \mathcal{O}_X(1)) = 0$.
2. $H^2(X, \mathcal{O}_X) = 0$, or X is Cohen-Macaulay and the canonical map
$$H^0(X, \omega_X(-1)) \otimes V^\vee \to H^0(X, \omega_X)$$
is surjective.

Then every formal deformation of X can be embedded into $\mathbb{P}^n$. This is easily seen by the proof above.

Another case where the deformations of X can be embedded into Z is given by the following result.

PROPOSITION 5.2.16. *Let $\mathcal{E}$ be a vector bundle of rank r on an n-dimensional manifold Z and $X \subseteq Z$ a closed subspace of codimension r which is the zero set of a section $\sigma \in H^0(Z, \mathcal{E})$. Then the following hold.*

1. If $H^{p+1}(Z, \Theta_Z \otimes \bigwedge^p \mathcal{E}^\vee) = 0$, $p \geq 0$, then every deformation of X is embeddable into Z.

2. If furthermore $H^p(Z, \mathcal{E} \otimes \bigwedge^p \mathcal{E}^\vee) = 0$, $p \geq 1$, then a versal deformation of X is given by the family

$$\mathfrak{X} := \{(z, \tau) \in Z \times H^0(Z, \mathcal{E}) \mid \tau(z) = 0\}$$

over $(S := H^0(Z, \mathcal{E}), \sigma)$.

PROOF. The Koszul complex

$$\mathcal{K}^\bullet : 0 \to \bigwedge^r \mathcal{E}^\vee \to \ldots \to \bigwedge^2 \mathcal{E}^\vee \to \mathcal{E}^\vee \xrightarrow{\sigma} \mathcal{O}_Z \to \mathcal{O}_X \to 0$$

is a locally free resolution of $\mathcal{O}_X$. Tensoring it with Θ_Z and using a simple spectral sequence argument yields that the condition in (1) implies the vanishing of $H^1(Z, \Theta_Z \otimes \mathcal{O}_X)$. By 2.3.6 this module is isomorphic to $\mathrm{Ex}(Z, \mathcal{O}_X)$ .Applying 5.2.11 every deformation of X can be embedded into Z.

Similarly, in case of (2) we can tensor the Koszul complex above with $\mathcal{E}$ and obtain with the same arguments as before that $H^1(Z, \mathcal{E} \otimes \mathcal{J}) = 0$, where $\mathcal{J} \subseteq \mathcal{O}_Z$ is the ideal sheaf of X. Thus (2) is a consequence of 5.2.10. □

Observe that the assumption in 5.2.16 (1) implies in particular that $H^1(Z, \Theta_Z) = 0$, i.e. Z is rigid. Similarly, the condition in (2) for $p = 1$ reads $H^1(Z, \mathcal{E} \otimes \mathcal{E}^\vee) = 0$. We will see in the next section that then $\mathcal{E}$ is rigid as a vector bundle.

COROLLARY 5.2.17. *Let $(Z, \mathcal{O}_Z(1))$ be a rigid projective manifold of dimension $\geq r + 2$ and $\mathcal{F}$ a vector bundle of rank r on Z such that $H^1(Z, \mathcal{F} \otimes \mathcal{F}^\vee) = 0$. Set $\mathcal{E} := \mathcal{F}(n)$ and consider a section $\sigma \in H^0(Z, \mathcal{E})$ such that the set of zeros $X := \{\sigma = 0\}$ has codimension r. Then the following hold.*

1. If $n \gg 0$ then the versal deformation of X is given by the subspace $\mathfrak{X} \subseteq Z \times S$ described in 5.2.16 (2). In particular, the versal deformation of X is smooth and all deformations are embeddable into Z.

2. The dimension of the base space of the semiuniversal deformation of X is given by

$$h^0(Z, \mathcal{E}) - h^0(Z, \mathcal{E} \otimes \mathcal{E}^\vee) + h^0(X, \Theta_X) - h^0(Z, \Theta_Z) \tag{a}$$

and the dimension of H_Z at $[X]$ by

$$h^0(Z, \mathcal{E}) - h^0(Z, \mathcal{E} \otimes \mathcal{E}^\vee). \tag{b}$$

PROOF. The groups

$$\begin{aligned} H^{p+1}\left(Z, \Theta_Z \otimes \textstyle\bigwedge^p \mathcal{E}^\vee\right) &\cong H^{p+1}\left(Z, \Theta_Z \otimes \left(\textstyle\bigwedge^p \mathcal{F}^\vee\right)(-np)\right), \quad p \geq 0, \\ H^p\left(Z, \mathcal{E} \otimes \textstyle\bigwedge^p \mathcal{E}^\vee\right) &\cong H^p\left(Z, \left(\mathcal{F} \otimes \textstyle\bigwedge^p \mathcal{F}^\vee\right)(-(p-1)n)\right), \quad p \geq 1, \end{aligned}$$

vanish for $n \gg 0$. Therefore (1) follows from 5.2.16. Moreover the proof of that result shows that $H^1(Z, \mathcal{E} \otimes \mathcal{J}) = 0$, where $\mathcal{J} \subseteq \mathcal{O}_Z$ is the ideal sheaf of X. As $\mathcal{H}om(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X) \cong \mathcal{E} \otimes \mathcal{O}_X$ there is an exact sequence

$$0 \to H^0(Z, \mathcal{E} \otimes \mathcal{J}) \to H^0(Z, \mathcal{E}) \to \mathrm{Hom}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X) \to 0.$$

Hence, in order to deduce (b) it is sufficient to show that the canonical map

$$H^0(Z, \mathcal{E} \otimes \mathcal{E}^\vee) \xrightarrow{1\otimes\sigma} H^0(Z, \mathcal{E} \otimes \mathcal{J})$$

is bijective. But this again follows by a simple spectral sequence argument applied to $\mathcal{E} \otimes \mathcal{K}^\bullet$, with $\mathcal{K}^\bullet$ the Koszul complex as in the proof of 5.2.16.

To establish (a) we consider the Kodaira-Spencer sequence

$$0 \to \mathrm{Der}(\mathcal{O}_X, \mathcal{O}_X) \to H^0(Z, \Theta_Z \otimes \mathcal{O}_X) \to \mathrm{Hom}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X) \to \mathrm{Ex}(X, \mathcal{O}_X) \to 0.$$

By the same argument as before $H^0(Z, \Theta_Z) \cong H^0(Z, \Theta \otimes \mathcal{O}_X)$. Hence, taking dimensions in this exact sequence and using (a), the desired formula follows. ☐

Remark 5.2.18. 1. Using the preceding corollary one can prove the following result, see [**Bor1, Bor2**]. Let $X = H_1 \cap \ldots \cap H_r$ be a complete intersection of dimension ≥ 2 in a homogeneous Kähler manifold $(Z, \mathcal{O}_Z(1))$ with $\mathrm{Pic}\, Z = \mathbb{Z}[\mathcal{O}_Z(1)]$, i.e. H_ϱ is a hypersurface defined by a section in some twist $\mathcal{O}_Z(d_\varrho)$. Then all deformations of X can be embedded into Z. Moreover, the basis of the semiuniversal deformation of X is smooth, and all deformations of X are again complete intersections in Z.

2. In the corollary above, the group $H^0(X, \Theta_X)$ can be non-zero, in general, even if $n \gg 0$ (cf. 2.5.15). However, if X is smooth then we will see in **??** that X has no non-trivial vector fields.

3. Let Z be a compact manifold and $X \subseteq Z$ a subscheme. Then it can happen that all deformations of X are embeddable although the vanishing criterion in **??** is not satisfied; for an example we refer the reader to [**Weh**, 3.9].

5.3. Deformations of modules

In this section we will consider deformations of modules on complex spaces which were introduced in 3.1.8 (2). We will compute the spaces of infinitesimal automorphism and infinitesimal deformations in homological terms. This will enable us to give criteria for when the basis of the semiuniversal deformation of a module is smooth. We also compute versal deformations by using extensions of modules.

Let $\pi : X \longrightarrow \Sigma$ be a morphism of complex spaces. We remind the reader that the deformation groupoid of modules consists of all pairs $(S, \mathcal{F})$ where $S \in \mathbf{An}_\Sigma$ and $\mathcal{F}$ is a coherent $\mathcal{O}_{X\times_\Sigma S}$-module that is S-flat. These pairs form a deformation theory $F \longrightarrow \mathbf{An}_\Sigma$ as explained in 3.1.8(3). We want to compute for a coherent $\mathcal{O}_S$-module $\mathcal{M}$ the spaces

$$\begin{aligned} \mathrm{Aut}_\Sigma(\mathcal{F}/S, \mathcal{M}) &:= \mathrm{Aut}_\Sigma(a/S, \mathcal{M}) \\ \mathrm{Ex}_\Sigma(\mathcal{F}/\mathcal{M}) &:= \mathrm{Ex}_\Sigma(a/S, \mathcal{M}), \end{aligned}$$

where $a := (S, \mathcal{F})$, see 3.3.1. In the following we write in brief $X_S := X \times_\Sigma S$ and denote by $\pi_S : X_S \longrightarrow S$ the projection.

Proposition 5.3.1. *There are canonical isomorphisms*

(1) $\mathrm{Aut}_\Sigma(\mathcal{F}/S, \mathcal{M}) \cong \mathrm{Hom}_{X_S}(\mathcal{F}, \mathcal{F} \otimes \pi_S^*(\mathcal{M}))$
(2) $\mathrm{Ex}_\Sigma(\mathcal{F}/S, \mathcal{M}) \cong \mathrm{Ext}^1_{X_S}(\mathcal{F}, \mathcal{F} \otimes \pi_S^*(\mathcal{M}))$.

Proof. Let $X_{S[\mathcal{M}]} := X_S \times_S S[\mathcal{M}]$ be the trivial extension of X_S by $\pi_S^*(\mathcal{M})$ so that $\mathcal{O}_{X_{S[\mathcal{M}]}} = \mathcal{O}_{X_S} \otimes \pi_S^*(\mathcal{M})\varepsilon$. By definition, an element of $\mathrm{Aut}_\Sigma(\mathcal{F}/S, \mathcal{M})$ is an automorphism β of $\mathcal{O}_{X_{S[\mathcal{M}]}}$-modules that fits into the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook] \arrow[d, "\mathrm{id}_{\mathcal{F}}"'] & \mathcal{F} \oplus \mathcal{F} \otimes \pi_S^*(\mathcal{M})\varepsilon \arrow[r, equal] \arrow[d, "\beta"'] & \mathcal{F} \otimes_{\mathcal{O}_{X_S}} \mathcal{O}_{X_{S[\mathcal{M}]}} \arrow[d, "\beta"'] \\
\mathcal{F} \arrow[r, hook] & \mathcal{F} \oplus \mathcal{F} \otimes \pi_S^*(\mathcal{M})\varepsilon \arrow[r, equal] & \mathcal{F} \otimes_{\mathcal{O}_{X_S}} \mathcal{O}_{X_{S[\mathcal{M}]}}.
\end{tikzcd}$$

Since β is $\mathcal{O}_{X_{S[\mathcal{M}]}}$ - linear it is uniquely determined by the $\mathcal{O}_{X_S}$-linear map

$$\beta|\mathcal{F} = \mathrm{id}_{\mathcal{F}} + \varepsilon\,\gamma \quad , \quad \gamma \in \mathrm{Hom}_{X_S}(\mathcal{F}, \mathcal{F} \otimes \pi_S^*(\mathcal{M})).$$

Clearly $\beta \longmapsto \gamma$ gives the desired bijection in (1).

For the proof of (2), observe first that an element in $\mathrm{Ex}_\Sigma(\mathcal{F}/S, \mathcal{M})$ is just the isomorphism class of an $\mathcal{O}_{X_{S[\mathcal{M}]}}$-module $\mathcal{F}'$ that is flat over $S[\mathcal{M}]$ and satisfies $\mathcal{F}'/\varepsilon\,\mathcal{F}' \cong \mathcal{F}$. This gives rise to an exact sequence of $\mathcal{O}_{X_{S[\mathcal{M}]}}$-modules

$$(*) \qquad \mathcal{O} \longrightarrow \mathcal{F} \otimes \pi_S(\mathcal{M}) \xrightarrow{\varepsilon} \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{O},$$

see 3.3.9. Considering this sequence as an extension of modules over $\mathcal{O}_{X_S} \hookrightarrow \mathcal{O}_{X_{S[\mathcal{M}]}}$, we get an element

$$E(\mathcal{F}') \in \mathrm{Ext}^1_{X_S}(\mathcal{F}, \mathcal{F} \otimes \pi_S^*(\mathcal{M})).$$

Conversely, given an extension of $\mathcal{O}_{X_S}$-modules as in $(*)$ we get an $\mathcal{O}_{X_{S[\mathcal{M}]}}$-structure on $\mathcal{F}'$, where the multiplication by ε is given as indicated. Using 3.3.9 $\mathcal{F}'$ is flat over $S[\mathcal{M}]$ and so defines an element in $\mathrm{Ex}_\Sigma(\mathcal{F}/S, \mathcal{M})$. It is clear from the construction that these maps are inverse to each other and so define the desired bijection in (2). □

Remark 5.3.2. In the special case that $\mathcal{F}$ is locally free on X_S the group $\mathrm{Ext}^1_{X_S}(\mathcal{F}, \mathcal{F} \otimes \pi_S^*(\mathcal{M}))$ is isomorphic to $H^1(X_S, \mathcal{E}nd\,(\mathcal{F}) \otimes \pi_S^*(\mathcal{M}))$. Thus we get a natural isomorphism

$$\mathrm{Ex}_\Sigma(\mathcal{F}/S, \mathcal{M}) \cong H^1(X_S, \mathcal{E}nd\,(\mathcal{F}) \otimes \pi_S^*(\mathcal{M})).$$

Assume that $\mathcal{F}_0$ is a coherent module on $X_0 := \pi^{-1}(0)$. If

$$\mathrm{Ex}_\Sigma(\mathcal{F}_0, \mathbb{C}) \cong \mathrm{Ext}^1_{X_0}(\mathcal{F}_0, \mathcal{F}_0)$$

is finite dimensional, then by Schlessingers theorem $\mathcal{F}_0$ admits a formal semiuniversal deformation. If $\mathcal{F}_0$ has compact support then $\mathrm{Ext}^1_{X_0}(\mathcal{F}_0, \mathcal{F}_0)$ is finite dimensional and a deep result due to [**TSi**] (see also [**BKo**]) shows that in this case there even is a convergent semiuniversal deformation.

Theorem 5.3.3. *Every coherent module $\mathcal{F}_0$ on X_0 with compact support admits a (convergent) semiuniversal deformation.*

We will now examine the question as to when the basis of the semiuniversal deformation of $\mathcal{F}_0$ is smooth.

Proposition 5.3.4. *Assume that $\mathrm{Ext}^1_{X_0}(\mathcal{F}_0, \mathcal{F}_0)$ is finite dimensional and that $\mathrm{Ext}^2_{X_0}(\mathcal{F}_0, \mathcal{F}_0) = 0$. Then the basis of the formally semiuniversal deformation of $\mathcal{F}_0$ is smooth over a closed subspace of Σ.*

In order to compute $\mathrm{Ext}^i_X(\mathcal{F}_0, \mathcal{F}_0)$ explicitly it is often useful to note that this group is isomorphic to $H^i(X, \mathcal{E}nd(\mathcal{F}_0))$ if $\mathcal{F}_0$ is locally free. We remark that, if $\mathcal{F}_0$ even admits a convergent semiuniversal deformation over (S,0), it follows that S is also smooth over a closed subspace of Σ. Before proving **??** we note the following corollary.

COROLLARY 5.3.5. *Assume that* $\Sigma = pt$ *is a simple point and that* $\mathcal{F}_0$ *is a coherent* $\mathcal{O}_X$*-module on the complex space* X *with* $\dim(\mathrm{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0)) < \infty$ *and* $\mathrm{Ext}^2_X(\mathcal{F}_0, \mathcal{F}_0) = 0$. *Then the basis of the formally semiuniversal deformation of* $\mathcal{F}_0$ *is smooth.*

PROOF OF 5.3.4. Let $\mathcal{F}$ be a deformation of $\mathcal{F}_0$ over an artinian base $\mathcal{S} = (\mathcal{S}, 0) \in \mathbf{An}_\Sigma$. By our criterion **??** we need to show that the functor

$$\mathbf{Coh}\, T \ni \mathcal{M} \longmapsto \mathrm{Ex}_\Sigma(\mathcal{F}/\mathcal{S}, \mathcal{M}) \cong \mathrm{Ext}^1_{X_\mathcal{S}}(\mathcal{F}, \mathcal{F} \otimes \pi^*_\mathcal{S}(\mathcal{M}))$$

is right exact. Using remark **??** it is sufficient to verify that $\mathrm{Ext}^2_{X_\mathcal{S}}(\mathcal{F}, \mathcal{F}_0) = 0$. Applying **??** it follows that

$$(*) \qquad \mathrm{Ext}^2_X(\mathcal{F}_0, \mathcal{F}_0) \longrightarrow \mathrm{Ext}^2_{X_\mathcal{S}}(\mathcal{F}, \mathcal{F}_0)$$

is surjective; note that $\mathrm{Tor}^X_p(\mathcal{F}, \mathcal{O}_X)$ vanishes for all $p \geq 1$. Since by assumption the left hand side of $(*)$ is zero, the result follows.

EXAMPLES 5.3.6. (1) If $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n}(d_r)$ then for $n \geq 2$ the group $\mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{F}, \mathcal{F})$ vanishes and so $\mathcal{F}$ is rigid. For $n = 1$ the number of moduli of $\mathcal{F}$ is given by the dimension of

$$\bigoplus_{i,j=1}^{r} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_i - d_j)).$$

Moreover, $\mathrm{Ext}^2_{\mathbb{P}^1}(\mathcal{F}, \mathcal{F})$ vanishes. Thus the basis of the semiuniversal deformation is always smooth.

(2) Suppose that Σ is a simple point and let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle. Then the dimension of extensions of $\mathcal{L}$ is given by the dimension $\mathrm{Ext}^1_X(\mathcal{L}, \mathcal{L}) \cong H^1(X, \mathcal{O}_X)$. Note that for a compact complex manifold $\mathrm{Pic}(X)$ is a complex Lie group of dimension $\dim H^1(X, \mathcal{O}_X)$. But in general $\mathrm{Ext}^2_X(\mathcal{O}_X, \mathcal{O}_X) \cong H^2(X, \mathcal{O}_X)$ does not vanish although $\mathrm{Pic}\, X$ is always smooth.

(3) Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^2$. Recall that $\mathcal{E}$ is said to be *simple* if $\mathrm{End}(\mathcal{E}) = \mathbb{C}$. From the above result it follows that the basis of a semiuniversal deformation of a simple bundle is always smooth. In fact,

$$\mathrm{Ext}^2_{\mathbb{P}^2}(\mathcal{E}, \mathcal{E}) \cong H^2(\mathbb{P}^2, \mathcal{E}nd\, \mathcal{E}) \cong H^0(\mathbb{P}^2, \mathcal{E}nd\, \mathcal{E}(-3))$$

by Serre duality. Moreover

$$\mathcal{E}nd\, \mathcal{E} \cong sl(\mathcal{E}) \oplus \mathcal{O}_X,$$

where $sl(\mathcal{E})$ are the endomorphisms of trace 0. Thus being simple means that $H^0(\mathbb{P}^2, sl(\mathcal{E})) = 0$ It follows that $H^0(\mathbb{P}^2, (\mathcal{E}nd\, \mathcal{E})(-3)) = 0$ if $\mathcal{E}$ is simple. The number of moduli can be computed from

$$\chi(\mathcal{E}nd\, \mathcal{E}) = h^0(\mathcal{E}nd\, \mathcal{E}) - h^1(\mathcal{E}nd\, \mathcal{E}) + h^2(\mathcal{E}nd\, \mathcal{E}) = r(c_1^2 - 2c_2) - c_1^2 + r^2,$$

where $c_j = c_j(\mathcal{E})$ are the Chern classes, see [**?**].

EXERCISE 5.3.7. Let X be a complex space, $x \in X$ and consider $\mathcal{M}_0 := \mathcal{O}_{X,x}/\mathfrak{m}_x$ as a module on X. Then the basis of a semiuniversal deformation of $\mathcal{M}_0$ is (X, x), and the module $\mathcal{M} = \mathcal{O}_{X \times X}/\mathcal{J}$ is the semiuniversal deformation of $\mathcal{M}_0$, where $\mathcal{J} \subseteq \mathcal{O}_{X \times X}$ is the ideal of the diagonal.

In particular, this example shows that a semiuniversal deformation of a module can have arbitrary singularities. Observe that $\mathcal{M}_0$ is the structure sheaf of the simple point $\{x\} \hookrightarrow X$. The reader may verify that the germ of the Douady space $(H_X, [\{x\}])$ is also isomorphic to (X, x).

It is sometimes useful to consider extension in order to compute deformation of modules. Let $X \to \Sigma$ as above and let $\mathcal{F}', \mathcal{F}''$ be fixed coherent $\mathcal{O}_X$-modules that are flat over Σ. Consider for a space $S \in \mathbf{An}_\Sigma$ all extension $(\mathcal{F}, \alpha, \beta)$

$$\mathcal{O} \longrightarrow \mathcal{F}'_S \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}''_S \longrightarrow 0,$$

where for a sheaf $\mathcal{G}$ on X we denote $\mathcal{G}_S$ the pullback of $\mathcal{G}$ under the map $X_S \longrightarrow X$. These extensions form a category, say $\mathbf{E}$, in an obvious way that is fibred over $\mathbf{An}_\Sigma$, so that the extensions form a deformation theory. Thus

$$(*) \qquad E(S) \cong \mathrm{Ext}^1_{X_S}(\mathcal{F}''_S, \mathcal{F}'_S).$$

Assigning to an extension $(\mathcal{F}, \alpha, \beta)$ the associated module $\mathcal{F}$ defines a functor $\mathbf{E} \to \mathbf{F}$ into the category of deformations of modules. For a coherent $\mathcal{O}_S$-module $\mathcal{M}$ let

$$\mathrm{Aut}_\Sigma((\mathcal{F}, \alpha, \beta)/\mathcal{S}, \mathcal{M}) \quad \text{and} \quad \mathrm{Ex}_\Sigma((\mathcal{F}, \alpha, \beta)/\mathcal{S}, \mathcal{M})$$

be the module of infinitesimal automorphism of the trivial extension of $a = (\mathcal{F}, \alpha, \beta)$ over $\mathcal{S}[\mathcal{M}]$ resp. the module of infinitesimal extensions.

Proposition 5.3.8. *There are canonical isomorphisms*

(1) $\mathrm{Aut}_\Sigma((\mathcal{F}, \alpha, \beta)/\mathcal{S}, \mathcal{M}) \cong \mathrm{Hom}_{X_S}(\mathcal{F}''_S, \mathcal{F}'_S \otimes \pi_S^*\mathcal{M})$
(2) $\mathrm{Ex}_\Sigma((\mathcal{F}, \alpha, \beta/S, \mathcal{M}) \cong \mathrm{Ext}^1_{X_S}(\mathcal{F}''_S, \mathcal{F}'_S \otimes \pi_S^*\mathcal{M})$.

Moreover, assigning to an extension the underlying module gives a map

$$\mathrm{Ex}_\Sigma((\mathcal{F}, \alpha, \beta)/\mathcal{S}, \mathcal{M}) \longrightarrow \mathrm{Ex}_\Sigma(\mathcal{F}/\mathcal{S}, \mathcal{M}),$$

and, under the identification in (2) resp in **??**, *this map identifies with the canonical map*

$$\mathrm{Ext}^1_{X_S}(\mathcal{F}''_S, \mathcal{F}'_S \otimes \pi_S^*\mathcal{M}) \longrightarrow \mathrm{Ext}^1_{X_S}(\mathcal{F}, \mathcal{F} \otimes \pi_S^*\mathcal{M})$$

induced by α and β.

Proof. By definition $\mathrm{Ex}_\Sigma((\mathcal{F}, \alpha, \beta)/\mathcal{S}, \mathcal{M})$ is the fibre of the map

$$E(\mathcal{S}[\mathcal{M}]) \longrightarrow E(\mathcal{S})$$

over the class of $(\mathcal{F}, \alpha, \beta)$ in $E(\mathcal{S})$. Because of $(*)$ this map can be identified with the map

$$\mathrm{Ext}^1_{X_{S[\mathcal{M}]}}(\mathcal{F}''_{S[\mathcal{M}]}, \mathcal{F}'_{S[\mathcal{M}]}) \xrightarrow{\gamma} \mathrm{Ext}^1_{X_S}(\mathcal{F}''_S, \mathcal{F}'_S).$$

The left hand side is isomorphic to

$$\mathrm{Ext}^1_{X_S}(\mathcal{F}''_S, \mathcal{F}'_{S[\mathcal{M}]}) \cong \mathrm{Ext}^1_{X_S}(\mathcal{F}''_S, \mathcal{F}'_S) \oplus \mathrm{Ext}^1_{X_S}(\mathcal{F}''_S, \mathcal{F}'_S \otimes \pi_S^*\mathcal{M}\varepsilon),$$

and the map γ is just the projection onto the first factor. This proves (2). Moreover, (1)follows along the same lines, since the automorphism of an extension $(\mathcal{F}, \alpha, \beta)$ in $E(S)$ can be identified in a natural way with the elements of $\mathrm{Hom}_{X_S}(\mathcal{F}''_S, \mathcal{F}'_S)$. □

Suppose now that $\Sigma = \{0\}$ is a simple point and

$$W \subseteq \mathrm{Ext}^1_X(\mathcal{F}'', \mathcal{F}')$$

is a subspace of finite dimension. Then we can form the tautological extension on $W \times X$

$$(*) \qquad \mathcal{O} \longrightarrow \mathcal{F}^1_W \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}''_W \longrightarrow 0$$

that is over each point $w \in W$ the extension given by $w \varepsilon \operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}')$. In more formal terms, W defines an element

$$\xi_W \varepsilon \operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}') \otimes W^\vee,$$

and since $W^\vee$ represents the (linear) functions on W ,the latter space can be canonically embedded into

$$\operatorname{Ext}^1_{X \times W}(\mathcal{F}''_W, \mathcal{F}'_W).$$

Thus ξ_w defines an extension as desired. It is clear that the Kodaira-Spencer map at each point of W can be identified with the given inclusion $W \subseteq \operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}')$. We will refer to $(*)$ as the *universal- extension* over W. In particular we obtain the following result.

Proposition 5.3.9. *Suppose that* $\dim_{\mathbb{C}} \operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}') < \infty$ *and consider the universal extension* $(\mathcal{F}, \alpha, \beta)$ *over* $X \times W$*, where* $W := \operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}')$*. Then* $(\mathcal{F}, \alpha, \beta)$ *is a semiuniversal deformation at each point of W.*

This result can be sometimes useful to get versal deformations of modules. Suppose that we are given a module $\mathcal{F}_0$ on a complex space X together with an extension

$$(*) \qquad \mathcal{O} \longrightarrow \mathcal{F}' \xrightarrow{\alpha} \mathcal{F}_0 \xrightarrow{\beta} \mathcal{F}'' \longrightarrow 0.$$

Let $p \varepsilon \operatorname{Ext}^1(\mathcal{F}'', \mathcal{F}')$ be the corresponding class. Then the following holds.

Corollary 5.3.10. *Suppose that the map induced by* α, β

$$\operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}') \longrightarrow \operatorname{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0)$$

is surjective. Let $W \subseteq \operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}')$ *be a finite dimensional subspace containing* p *and mapping bijectively onto* $\operatorname{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0)$*. Then the universal extension*

$$\mathcal{O} \longrightarrow \mathcal{F}'_W \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}''_W \longrightarrow 0$$

defines a module which is a formally semiuniversal deformation of $\mathcal{F}_0$ *at* p*. In particular, the formally semiuniversal deformation of* $\mathcal{F}_0$ *has a smooth base space.*

Note that if $\mathcal{F}_0$ admits a versal deformation then the universal extension is semiuniversal (see **??**).

Example 5.3.11. Let us consider

$$\mathcal{F}_0 := \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(n)$$

and the trivial extension

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(n) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(n) \longrightarrow 0$$

on $\mathbb{P}^1$. Then the induced map

$$\operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}) \longrightarrow \operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{F}_0, \mathcal{F}_0)$$

is bijective. Hence a versal deformation of $\mathcal{F}_0$ is given by the universal extension, and this universal extension even defines the semiuniversal deformation of $\mathcal{F}_0$. The dimension of the base space equals

$$\dim \operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}) = \dim H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-n)) = \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2)) = n-1.$$

The reader may verify that by this deformation $\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(n)$ can be deformed only into the bundles $\mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathcal{O}_{\mathbb{P}^1}(b)$, where $0 \leq a \leq b \leq n$ and $a + b = n$.

EXERCISE 5.3.12. Consider $W := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2))$ and the map

$$\alpha = (\tau, x^{n-1}, y^{n-1}) : \mathcal{O}_{\mathbb{P}^1 \times W}(2-n) \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times W} \otimes \mathcal{O}_{\mathbb{P}^1 \times W}(1)^{\oplus 2},$$

where $[x : y]$ denote the coordinates on $\mathbb{P}^1$ and τ is the map corresponding to the tautological section in $H^0(\mathcal{O}_{\mathbb{P}^1 \times W}(n-2))$, i.e. $\tau(w, [x:y]) = w(x:y)$. Show that a semiuniversal deformation of $\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(n)$ is given by $\mathcal{F} := \operatorname{Coker} \alpha$, and that a semiuniversal extension is given by

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times W} \xrightarrow{\beta} \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times W}(n) \longrightarrow 0,$$

where β is the composition of the inclusion map $\mathcal{O}_{\mathbb{P}^1 \times W} \hookrightarrow \mathcal{O}_{\mathbb{P}^1 \times W} \oplus \mathcal{O}_{\mathbb{P}^1 \times W}(1)^{\oplus 2}$ and the projection $\mathcal{O}_{\mathbb{P}^1 \times W} \oplus \mathcal{O}_{\mathbb{P}^1 \times W}(1)^{\oplus 2} \to \mathcal{F}$.

In general we have the following simple criterion.

EXERCISE 5.3.13. (a) Let X be a complex space and $\mathcal{F}', \mathcal{F}''$ coherent $\mathcal{O}_X$-modules and

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}'' \longrightarrow 0$$

an extension. Suppose that the following conditions are satisfied.

(1) $\operatorname{Ext}^1_X(\mathcal{F}', \mathcal{F}') = \operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}'') = 0$;
(2) $\operatorname{Ext}^1_X(\mathcal{F}', \mathcal{F}'') = 0$;
(3) $\operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}')$ has a finite dimension.

Then the map $\operatorname{Ext}^1_X(\mathcal{F}'', \mathcal{F}') \longrightarrow \operatorname{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0)$ is surjective and so 5.3.10 applies.

(b) Assume that $\mathcal{F}', \mathcal{F}''$ are line bundles with $H^1(X, \mathcal{F}'^{\vee} \otimes \mathcal{F}'') = 0$ and that $H^1(X, \mathcal{O}_X) = 0$. Check that then (1) -(3) in (a) are satisfied.

In the case of deformations of locally free sheaves we can improve the smoothness criterion 5.3.5 as follows. For simplicity, we restrict to the case that Σ is a simple point.

THEOREM 5.3.14. *Let X be a compact Khler manifold and $\mathcal{F}_0$ a locally free sheaf on X. Assume that $H^2(X, sl(\mathcal{F}_0)) = 0$, where $sl(\mathcal{F}_0) \subseteq \mathcal{E}nd\,\mathcal{F}_0$ is the subsheaf of traceless endomorphism. Then the basis of a versal deformation of $\mathcal{F}_0$ is smooth.*

PROOF. Using **??** we need to show that for every deformation, say, $\mathcal{F}$ of $\mathcal{F}_0$ over an artinian base $\mathcal{S} = (\mathcal{S}, 0)$ the functor

$$\mathbf{Coh}(\mathcal{S}) \ni \mathcal{M} \longmapsto \operatorname{Ext}^1_{X_{\mathcal{S}}}(\mathcal{F}, \mathcal{F} \otimes \pi^*_{\mathcal{S}}\mathcal{M}) \cong H^1(X_{\mathcal{S}}, (\mathcal{E}nd\,\mathcal{F}) \otimes \pi^*_{\mathcal{S}}\mathcal{M})$$

is right exact. Equivalently, we will verify that the functors

$$\mathcal{M} \longmapsto H^1(X_S, sl(\mathcal{F}) \otimes \pi^*_S\mathcal{M}) \quad \text{and} \quad \mathcal{M} \longmapsto H^1(X_S, \mathcal{O}_X \otimes \pi^*_S\mathcal{M})$$

are right exact. But this again follows for the first functor from **??** (applied to $k = 2$, $\mathcal{F} = \mathcal{O}_{X_S}$ and $\mathcal{G} = sl(\mathcal{F})$) resp. is contained in **??** □

EXAMPLE 5.3.15. (1) Assume that X is a compact Khler surface with $\omega_X \cong \mathcal{O}_X$ and that $\mathcal{F}_0$ is a simple vector bundle on X. Then the basis of the semiuniversal deformation of $\mathcal{F}_0$ is smooth. This follow from the fact that $H^2(X, sl(\mathcal{E}))$ is dual to $H^0(X, sl(\mathcal{E}))$ and so vanishes.

(2) Note that the result above also applies to line bundles $\mathcal{L}$ on Khler manifolds in which case $sl(\mathcal{L}) = 0$. Thus we obtain a new argument for that the Picard-variety is smooth.

In the remaining part of this section we will treat deformations of modules on singularities. Let $\pi : (X, 0)\longrightarrow(\Sigma, 0)$ be a fixed holomorphic map of complex space germs. For a space $(S, 0) \in \mathbf{An}_{\Sigma,0}$ we consider S-flat modules $\mathcal{M}$ on $(X_S, 0)$ where again $X_S := X \times_\Sigma S$. These modules form a deformation theory $\mathbf{F}\longrightarrow \mathbf{An}_{S,0}$ as before. In analogy with 5.3.1 we have the following result.

Proposition 5.3.16. *Let* $a = ((S, 0), \mathcal{F}) \in \mathbf{F}(S, 0)$. *Then for* $\mathcal{M} \in \mathbf{Coh}(S, 0)$ *there are canonical isomorphisms.*

$$\mathrm{Aut}_{\Sigma,0}(\mathcal{F}/S, \mathcal{M}) \cong \mathcal{H}om_{X_S}(\mathcal{F}, \mathcal{F} \otimes \pi^*\mathcal{M})_0$$
$$\mathrm{Ex}_{\Sigma,0}(a/S, \mathcal{M}) \cong \mathcal{E}xt^1_{X_S}(\mathcal{F}, \mathcal{F} \otimes \pi^*\mathcal{M})_0,$$

where the index 0 denotes the stalk at 0 and $\pi : X_S\longrightarrow S$ *is the projection.*

The *proof* is the same is the same as that of 5.3.1. In particular it follows that a module $\mathcal{F}_0$ on $X_0 := \pi^-1(0)$ admits a formally semiuniversal deformation provided that $\mathcal{E}xt^1_{X_0}(\mathcal{F}_0, \mathcal{F}_0)_0$ has finite dimension. This, for instance, is the case if $\mathcal{F}_0$ has an isolated singularity at 0, i.e. $\mathcal{F}_0$ is locally free on $X_0\{0\}$. More generally, the following deep result holds.

Theorem 5.3.17. *(*[**Tra**], [**BKo**]*) If* $\mathcal{E}xt^1_{X_0}(\mathcal{F}_0, \mathcal{F}_0)_0$ *has finite dimension then* $\mathcal{F}_0$ *admits a (convergent) semiuniversal deformation.*

In the following, let us always assume that $\mathcal{E}xt^1_{X_0}(\mathcal{F}_0, \mathcal{F}_0)_0$ has finite dimension. Concerning smoothness, we have –with the same proof– the analogous result as in 5.3.4.

Proposition 5.3.18. *If the vector space* $\mathcal{E}xt^1_{X_0}(\mathcal{F}_0, \mathcal{F}_0)_0$ *has finite dimension and if* $\mathcal{E}xt^2_{X_0}(\mathcal{F}_0, \mathcal{F}_0)_0 = 0$ *then the basis of the semiuniversal deformation of* $\mathcal{F}_0$ *is smooth.*

Examples 5.3.19. (1) If $\mathrm{pdim}_\mathbb{C}\, \mathcal{F}_0 \leq 1$, i.e. $\mathcal{F}_0$ admits a resolution

$$(*) \qquad 0\longrightarrow\mathcal{O}^m_{X_0} \xrightarrow{\alpha} \mathcal{O}^n_{X_0}\longrightarrow\mathcal{F}_0\longrightarrow 0$$

then the above $\mathcal{E}xt^2$-group vanishes and so 5.3.18 applies. More concretely, the module $\mathcal{E}xt^1_{X_0}(\mathcal{F}_0, \mathcal{F})_0$ is the cokernel of the map

$$\alpha^\vee : \mathrm{Hom}(A^n, F_0)\longrightarrow \mathrm{Hom}(A^m, F_0),$$

where $A := \mathcal{O}_{X_0},0$ and F_0 is the stalk of $\mathcal{F}_0$ at 0. Take vectors $\mathcal{Z}_1, \dots, \mathcal{Z}_k \in \mathrm{Hom}(A^m, F_0)$ whose residue classes generate $\mathrm{Ext}^1_A(F_0, F_0)$ as a vector space, and consider matrices $M_i \in \mathrm{Hom}(A^m, A^n)$ mapping onto $\mathcal{Z}_i$ in $\mathrm{Hom}(A^m, F_0)$ under the map induced by $A^n\longrightarrow F_0$. Now the versal deformation can be described in algebraic terms as follows. Let $B = A\{t_1, \dots, t_k\}$ be the free power series ring over B and F the cokernel of the map

$$B^m\longrightarrow B^n,$$

where the map is given by $M := \alpha + \sum t_iM_i$. Observe that by [**Mat**]

$$0\longrightarrow B^m\longrightarrow B^n\longrightarrow F\longrightarrow 0$$

is exact and that this sequence when tensored with A gives the original sequence $(*)$. In particular F is flat over $\mathbb{C}\{t_1, \dots, t_k\}$. The reader may verify that the sheaf on $(X_0 \times \mathbb{C}^k, 0)$ associated to F is the semiuniversal deformation of $\mathcal{F}_0$.

(2) Let $(X_0, 0)$ be a Cohen-Macaulay singularity and ω_{X_0} the dualizing module. Then $\mathcal{E}xt^i_{X_0}(\omega_{X_0},\omega_{X_0}) = 0$ for $i \geq 1$ (see**??**) and so ω_{X_0} is rigid.

Exercise 5.3.20. (1) Let $f = g \cdot h$ be a reducible nonzero element in the maximal ideal of $\mathbb{C}\{t_1, \dots, t_n\}$ and let $(X, 0)$ be the analytic germ defined by $\mathcal{O}_{X,0} = \mathbb{C}\{t_1, \dots, t_n\}/(f)$. Set $\mathcal{F}_0 = \mathcal{O}_{X,0}/(g)$.

(a) Show that

$$\dots \xrightarrow{g} \mathcal{O}_{X,0} \xrightarrow{h} \mathcal{O}_{X,0} \xrightarrow{g} \mathcal{O}_{X,0} \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

is a free resolution of $\mathcal{F}_0$ as $\mathcal{O}_{X,0}$-module.

(b) Show that $\mathcal{E}xt^1_X(\mathcal{F}_0, \mathcal{F}_0)_0 = 0$ if and only if g and h are relatively prime.

(c) Show that $\mathcal{E}xt^2_X(\mathcal{F}_0, \mathcal{F}_0)_0 \cong \mathcal{O}_{X,0}/(g, h)\mathcal{O}_{X,0}$.

(2) For which i, n is $\Omega^i_{\mathbb{P}^n}$ rigid on $\mathbb{P}^n$?

5.4. On the dimension of the base space of the semiuniversal deformation

In this section we will generalize th smoothness criterion given in 5.1.1. In general the Ex-functor considered there is not exact. The purpose of this section is to introduce a certain number, say k, measuring to what exact this functor is not exact. Our main result is that the dimension of the base space of the semiuniversal deformation has dimension at least

$$\dim \mathrm{Ex}(a_0, \mathbb{C}) - k\ .$$

We remark that later on we will introduce obstruction theories associating to every deformation a over a base S an $\mathcal{O}_S$–module $\mathrm{ob}(a, \mathcal{O}_S)$, and to every extension of S by $\mathcal{O}_S$ an element of this module which vanishes iff the deformation a admits an extension. We will show that with such obstruction theories we have even that the basis of the formally semiuniversal deformation is given as a subspace of $(\Sigma \times \mathbb{C}^n, 0)^\wedge$ by at most $k = \dim_{\mathbb{C}} \mathrm{ob}(a_0, \mathbb{C})$ equations, where $n = \dim_{\mathbb{C}} \mathrm{Ex}(a_0, \mathbb{C})$ is the number of infinitesimal deformations.

In practise it is cumbersome to construct effective obstruction theories. The advantage of the main result of this section is that only exactness properties of the Ex–functors are needed.

Let $p : \mathbf{F} \to \mathbf{An}$ be a deformation theory and let $a_0 \in \mathbf{F}(0)$ be an element with $\mathrm{Ex}(a_0, \mathbb{C})$ finite dimensional so that a_0 admits a semiuniversal deformation $\bar{b}$ over some base $(\bar{T}, 0)$.

The main result in the following theorem.

Theorem 5.4.1. *Assume that for every deformation a over an artinian base S and every exact sequence of $\mathcal{O}_S$–modules $0 \to \mathcal{O}_S/\mathfrak{m}_S \to \mathcal{M} \to \mathcal{M}' \to 0$ the cokernel of the map*

$$\mathrm{Ex}(a/S, \mathcal{M}) \to \mathrm{Ex}(a/S, \mathcal{M}'')$$

has dimension at most k. Then the basis $(\bar{T}, 0)$ of the formally semiuniversal deformation of a_0 has dimension at least

$$\dim_{\mathbb{C}} \mathrm{Ex}(a_0, \mathbb{C}) - k\ .$$

The essential tool for the proof is the following lemma.

Lemma 5.4.2. *Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be an exact sequence of $\mathcal{O}_{\bar{T}}$–modules and*

$$\mathrm{Hom}_{\bar{T}}\left(\Omega^1_{\bar{T}}, \mathcal{M}''\right) \cong \mathrm{Der}\left(\mathcal{O}_{\bar{T}}, \mathcal{M}''\right) \xrightarrow{\partial} \mathrm{Ext}\left(\Omega^1_{\bar{T}}, \mathcal{M}'\right)$$

be the boundary map in the associated long Ext-sequence. Then the following hold.

(1) *There is factorization*

$$\begin{tikzcd}
\operatorname{Der}\left(\mathcal{O}_{\bar{T}}, \mathcal{M}''\right) \arrow[rr, "\partial"] \arrow[dr, "\delta_{KS}"'] & & \operatorname{Ext}^1_{\bar{T}}\left(\Omega^1_{\bar{T}}, \mathcal{M}'\right) \\
& \operatorname{Ex}\left(\bar{b}/\bar{T}, \mathcal{M}''\right) \arrow[ur, dotted] &
\end{tikzcd}$$

(2) *There is an exact sequence of artinian modules* $0 \to \mathbb{C} \to \mathcal{M} \to \mathcal{M}'' \to 0$ *such that* ∂ *becomes surjective.*

Proof. For the proof of (1) observe first that δ_{KS} is surjective by the versality of $\bar{b}$. Therefore and in view of the Kodaira-Spencer sequence it suffices to show that ∂ vanishes on the image of the map β'' in the following commutative diagram

$$\begin{tikzcd}
\operatorname{Aut}(\bar{b}, \mathcal{M}'') \arrow[r, "\beta''"] \arrow[d] & \operatorname{Der}\left(\mathcal{O}_{\bar{T}}, \mathcal{M}''\right) \arrow[r, "\delta_{KS}"] \arrow[d, "\partial'"'] & \operatorname{Ex}(\bar{b}/\bar{T}, \mathcal{M}'') \arrow[r] & 0 \\
\operatorname{Ex}(\bar{b}, \mathcal{M}') \arrow[r] & \operatorname{Ex}(\bar{T}, \mathcal{M}').
\end{tikzcd}$$

Again by versality $\operatorname{Ex}(\bar{b}, \mathcal{M}') = 0$. Therefore ∂' vanishes on the image of β''. Since there is an factorization of ∂' as

$$\operatorname{Der}\left(\mathcal{O}_{\bar{T}}, \mathcal{M}''\right) \xrightarrow{\partial} \operatorname{Ext}^1_{\bar{T}}\left(\Omega^1_{\bar{T}}, \mathcal{M}'\right) \overset{j}{\hookrightarrow} \operatorname{Ex}(\bar{T}, \mathcal{M}')$$

with j being injective, see **??**, this implies that ∂ vanishes on image of β'' too.

For the proof of (2) set $\mathcal{O}_{T_n} = \mathcal{O}_{\bar{T}}/\mathfrak{m}_{\bar{T}}^{n+1}$ and denote by $\mathfrak{m}_n \subseteq \mathcal{O}_{T_n}$ the maximal ideal. Dualizing $\mathfrak{m}_n$, $\mathcal{O}_{T_n}$ as vector spaces over $\mathbb{C}$ gives a sequence of $\mathcal{O}_{\bar{T}}$–modules

$$0 \to \mathbb{C} \to \mathcal{O}_{T_n}^{\vee} \to \mathfrak{m}_n^{\vee} \to 0.$$

The module $\mathcal{O}_{T_n}^{\vee}$ is an injective $\mathcal{O}_{T_n}$-module since

$$M \longmapsto \operatorname{Hom}_{\mathcal{O}_{T_n}}(M, \mathcal{O}_{T_n}^{\vee}) \cong \operatorname{Hom}_{\mathbb{C}}(M, \mathbb{C})$$

is an exact functor on the $\mathcal{O}_{T_n}$-modules. Hence $\operatorname{Ext}^1_{T_n}\left(\Omega^1_{T_n}, \mathcal{O}_{T_n}^{\vee}\right) = 0$ and so the map ∂_n in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Der}\left(\mathcal{O}_{\bar{T}}, \mathfrak{m}_n^{\vee}\right) \arrow[r, "\partial"] & \operatorname{Ext}^1_{\bar{T}}\left(\Omega^1_{\bar{T}}, \mathbb{C}\right) \arrow[r] & \cdots \\
\cdots \arrow[r] & \operatorname{Der}\left(\mathcal{O}_{T_n}, \mathfrak{m}_n^{\vee}\right) \arrow[r, "\partial_n"] \arrow[u] & \operatorname{Ext}^1_{T_n}\left(\Omega^1_{T_n}, \mathbb{C}\right) \arrow[r] \arrow[u, "\gamma_n"'] & 0
\end{tikzcd}$$

is surjective. As γ_n is surjective for $n \gg 0$ also ∂ has to be surjective. □

Proof of 5.4.2. Let $T_n \subset \bar{T}$ be the n^{th} infinitesimal neighbourhood, $\bar{b}$ the semiuniversal deformation of a_0, and $b_n := b|T_n$. Write $\mathcal{O}_{\bar{T}} = R/I$ with $R := \mathbb{C}[[X_1, \ldots, X_d]]$, where $d := \dim \operatorname{Ex}(a_0, \mathbb{C})$ so that $I \subseteq \mathfrak{m}_R^2$.

Using **??** it suffices to prove that

$$(*) \qquad \dim \operatorname{Ext}^1_{\bar{T}}\left(\Omega^1_{\bar{T}}, \mathbb{C}\right) \leq k \ .$$

By 5.4.2 (2) there is an exact sequence of artinian modules $0 \to \mathbb{C} \to \mathcal{M} \to \mathcal{M}'' \to 0$ such that the map ∂ in the commutative diagram

$$\begin{tikzcd}
\mathrm{Der}\,(\mathcal{O}_{\bar{T}}, \mathcal{M}) \arrow[r] \arrow[d, "\delta_{KS}"'] & \mathrm{Der}\,(\mathcal{O}_{\bar{T}}, \mathcal{M}'') \arrow[r, "\partial"] \arrow[d, "\delta_{KS}"'] & \mathrm{Ext}^1_{\bar{T}}\left(\Omega^1_{\bar{T}}, \mathbb{C}\right) \arrow[r] & 0 \\
\mathrm{Ex}\,\left(\bar{b}/\bar{T}, \mathcal{M}\right) \arrow[r, "\gamma"'] & \mathrm{Ex}\,\left(\bar{b}/\bar{T}, \mathcal{M}''\right) \arrow[ur, dotted] & &
\end{tikzcd}$$

is surjective. By 5.4.2 (1) the map ∂ can be factored as shown by the dotted arrow. Since $\bar{b}$ is the semiuniversal deformation, the Kodaira-Spencer maps are surjective. A simple diagram chasing shows that $\mathrm{coker}\,\gamma$ is isomorphic to $\mathrm{Ext}^1_{\bar{T}}\left(\Omega^1_{\bar{T}}, \mathbb{C}\right)$. Using the assumption that $\mathrm{coker}\,\gamma$ has dimension $\leq k$, $(*)$ follows. □

In the case of normal compact complex spaces we obtain the following estimate.

Proposition 5.4.3 (Deformations of compact spaces). *Let X be a normal space and $(S, 0)$ the basis of the semiuniversal deformation of X. Then*

$$\dim_0 S \geq \dim_{\mathbb{C}} \mathrm{Ext}^1_X(\Omega^1_X, \mathcal{O}_X) - \mathrm{Ext}^2_X(\Omega^1_X, \mathcal{O}_X).$$

Proof. Let $\pi : \mathfrak{X} \to T$ be a deformation of X over an artinian base. Uisng the above criterion and the identification

$$\mathrm{Ex}(\mathfrak{X}/T, \mathcal{M}) \cong \mathrm{Ext}^1_{\mathfrak{X}}(\Omega^1_{\mathfrak{X}/T}, \pi^*\mathcal{M}),$$

we need to prove that for an exact sequence $0 \to \mathcal{O}_T/\mathfrak{m}_T \to \mathcal{M} \to \mathcal{M}'' \to 0$ the induced map β in the cohomology sequence

$$(*) \qquad \ldots \to \mathrm{Ext}^1_{\mathfrak{X}}(\Omega^1_{\mathfrak{X}/T}, \pi^*\mathcal{M}) \xrightarrow{\beta} \mathrm{Ext}^1_{\mathfrak{X}}(\Omega^1_{\mathfrak{X}/T}, \pi^*\mathcal{M}'') \to \mathrm{Ext}^2_{\mathfrak{X}}(\Omega^1_{\mathfrak{X}/T}, \mathcal{O}_X)$$

has a cokernel of dimension at most $k := \dim_{\mathbb{C}} \mathrm{Ext}^1_X(\Omega^1_{X/T}, \mathcal{O}_X)$. It follows from 5.1.8 that the canonical map

$$\mathrm{Ext}^2_X(\Omega^1_{X/T}, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^2_{\mathfrak{X}}(\Omega^1_{\mathfrak{X}/T}, \mathcal{O}_X)$$

is surjective, i.e. $\mathrm{Ext}^2_{\mathfrak{X}}(\Omega^1_{\mathfrak{X}/T}, \mathcal{O}_X)$ has dimension $\leq k$. In view of the exact sequence $(*)$ this proves the result. □

We can state a similar result for locall trivial deformations. The proof follows again from 5.4.1 since the infinitesimal deformations are given by $H^1(X, \Theta_X)$ and the number k in loc.cit can be estimated by $h^2(X, \Theta_X)$.

Proposition 5.4.4 (Locally trivial deformations). *Let X be a compact complex space and $(S, 0)$ the basis of the formally semiuniversal locally trivial deformation of X. Then*

$$\dim_0 S \geq h^1(X, \Theta_X) - h^2(X, \Theta_X).$$

The special case of complex manifolds is of particular importance and so we state it in the following corollary.

Corollary 5.4.5 (Deformations of compact manifolds). *Let X be a compact complex manifold and $(S, 0)$ the basis of the semiuniversal deformation of X. Then*

$$\dim_0 S \geq h^1(X, \Theta_X) - h^2(X, \Theta_X).$$

In the same way we can deduce the following two applications to deformations of subspaces and modules; the proofs are analogeous to the proof of 5.4.3 and left to the reader.

Proposition 5.4.6 (Deformations of subspaces). *Let $X \subseteq Z$ be a compact subspace with ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_Z$ and $p = [X] \in H_Z$ the associated point in the Douady space. Assume that the following condition is satisfied.*

(*) $\operatorname{grade}_T \mathcal{O}_X \geq 1$, *where T denotes the analytic set of points where $\mathcal{J}$ is not locally generated by a regular sequence.*

Then

$$\dim_p H_Z \geq \dim_{\mathbb{C}} \operatorname{Hom}_X(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X) - \dim_{\mathbb{C}} \operatorname{Ext}^1_X(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X).$$

Proposition 5.4.7 (Deformations of modules). *Let X be a complex space and $\mathcal{F}$ a coherent sheaf on X with compact support. Then the basis of the semiuniversal deformation of $\mathcal{F}_0$ has dimension at least*

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F}) - \dim_{\mathbb{C}} \operatorname{Ext}^2_X(\mathcal{F}, \mathcal{F}).$$

Remark 5.4.8. There is also a relative version of 5.4.1 as follows. Let $p : \mathbf{F} \to \mathbf{An}_{(\Sigma,0)}$ be a deformation theory and $a_0 \in \mathbf{F}(0)$, where 0 denotes the reduced point. Assume that for every deformation a over an artinian base $S \in \mathbf{An}_{(\Sigma,0)}$ and every exact sequence of $\mathcal{O}_S$–modules $0 \to \mathcal{O}_S/\mathfrak{m}_S \to \mathcal{M} \to \mathcal{M}' \to 0$ the cokernel of the map

$$\operatorname{Ex}_{(\Sigma,0)}(a/S, \mathcal{M}) \to \operatorname{Ex}_{(\Sigma,0)}(a/S, \mathcal{M}'')$$

has dimension at most k. Then the basis $(\bar{T}, 0)$ of the formally semiuniversal deformation of a_0 has dimension at least $\dim_{\mathbb{C}} \operatorname{Ex}_{(\Sigma,0)}(a_0, \mathbb{C}) - k$.

One might be tempted to ask whether the stronger estimate

$$(Q) \qquad \dim_0 \bar{T} \geq \dim_0 \Sigma + \dim_{\mathbb{C}} \operatorname{Ex}_{(\Sigma,0)}(a_0, \mathbb{C}) - k$$

holds. However, this is not true in general. For instance, take the embedded deformations of a point, say, x in some space $X \in \mathbf{An}_{(\Sigma,0)}$. Then the versal deformation is given by (X, x) (see **??**) whereas in general the estimate (Q) is not true. For instance, the most striking example is given by $X := \{x\}$ a reduced point.

5.5. Deformations of complexes and applications

Deformations of complexes. Let $\mathcal{K}^j$, $j \in \mathbb{Z}$, be a fixed family of coherent modules on a given complex space X. To these data we can associate the following deformation groupoid $p : \mathbf{E} \to \mathbf{An}$: an object over S is given by a complex of $\mathcal{O}_{X \times S}$–modules

$$(\mathcal{K}^\bullet_S, \partial)\,,$$

where $\mathcal{K}^j_S := p_1^*(\mathcal{K}^j)$ with $p_1 : X \times S \to X$ the projection. Thus such an object corresponds uniquely to a morphism $\partial : \mathcal{K}^\bullet_S \to \mathcal{K}^\bullet_S$ of degree 1 satisfying $\partial^2 = 0$. Moreover the morphisms in $\mathbf{E}$ are given by morphisms of complexes. Using **??** it follows that this constitutes a deformation theory.

It is easy to describe the modules of infinitesimal automorphisms and deformations

$$\operatorname{Aut}((\mathcal{K}^\bullet, \partial)/S, \mathcal{M}) \quad \text{and} \quad \operatorname{Ex}((\mathcal{K}^\bullet, \partial), \mathcal{M})$$

for this deformation theory. Let $p_2 : X \times S \to S$ be the projection onto S and let

$$H^p\,(\operatorname{Hom}(\mathcal{K}^\bullet_S, \mathcal{K}^\bullet_S \otimes p_2^*\mathcal{M}))$$

denote the cohomology of the complex $\operatorname{Hom}(\mathcal{K}^\bullet_S, \mathcal{K}^\bullet_S \otimes p_2^*\mathcal{M})$. Recall that an element of $H^p\,(\operatorname{Hom}(\mathcal{K}^\bullet_S, \mathcal{K}^\bullet_S \otimes p_2^*\mathcal{M}))$ is just the homotopy class of a morphism of complexes $\mathcal{K}^\bullet_S \to \mathcal{K}^\bullet_S \otimes p_2^*\mathcal{M}$ of degree p.

PROPOSITION 5.5.1. *There are canonical isomorphisms*

(1) $\mathrm{Aut}\left((\mathcal{K}_S^\bullet,\partial),\mathcal{M}\right)\cong H^0\left(\mathrm{Hom}\left(\mathcal{K}_S^\bullet,\mathcal{K}_S^\bullet\otimes p_2^*\mathcal{M}\right)\right)$.
(2) $\mathrm{Ex}\left((\mathcal{K}_S^\bullet,\partial),\mathcal{M}\right)\cong H^1\left(\mathrm{Hom}\left(\mathcal{K}_S^\bullet,\mathcal{K}_S^\bullet\otimes p_2^*\mathcal{M}\right)\right)$.

PROOF. For the proof of (1) let

$$\varphi:\mathcal{K}^\bullet_{S[\mathcal{M}]}\longrightarrow\mathcal{K}^\bullet_{S[\mathcal{M}]}$$

be an infinitesimal automorphism of

$$\mathcal{K}^\bullet_{S[\mathcal{M}]}=\mathcal{K}_S^\bullet\oplus\mathcal{K}_S^\bullet\otimes p_2^*\mathcal{M}\cdot\varepsilon,$$

where we equip this complex with the differential $\bar\partial=\partial+\partial\otimes\mathrm{id}_\mathcal{M}\cdot\varepsilon$. This automorphism can be written as $\varphi=\mathrm{id}+\varphi_2\cdot\varepsilon$ with a morphism $\varphi_2:\mathcal{K}_S^\bullet\to\mathcal{K}_S^\bullet\otimes p_2^*\mathcal{M}$. The reader may easily verify that the equation $\bar\partial\varphi=\varphi\bar\partial$ is equivalent to $(\partial\otimes\mathrm{id}_\mathcal{M})\varphi_2=\varphi_2\partial$. Thus associating to φ the map φ_2 gives the desired bijection in (1).

For the proof of (2) let $\bar\partial$ be a differential on $\mathcal{K}^\bullet_{S[\mathcal{M}]}$ with $\bar\partial\equiv\partial \bmod \varepsilon$. We can write $\bar\partial=\partial+\psi\cdot\varepsilon$ with a morphism

$$\psi:\mathcal{K}_S^\bullet\longrightarrow\mathcal{K}_S^\bullet\otimes p_2^*\mathcal{M}$$

of degree 1. The condition $\bar\partial^2=0$ is equivalent to $(\partial\otimes\mathrm{id}_\mathcal{M})\psi+\psi\partial=0$. Hence ψ defines an element in $H^1\left(\mathrm{Hom}(\mathcal{K}_S^\bullet,\mathcal{K}_S^\bullet\otimes p_2^*\mathcal{M})\right)$. The reader may easily check that this gives a bijection as required in (2). ☐

5.5.2. Assume now that $\mathcal{K}^i=0$ for $i>0$ and that there is given a structure as a complex $\mathcal{K}^\bullet=(\mathcal{K}^\bullet,\partial)$ on X such that

$$H^p(\mathcal{K}^\bullet)=\begin{cases}\mathcal{F}_0 & ,\ p=0\\ 0 & ,\ p\neq 0\ .\end{cases}$$

In particular $\mathcal{F}_0$ is a quotient of $\mathcal{K}^0$. Now let us consider a deformation

$$\mathcal{K}_S^\bullet=(\mathcal{K}_S^\bullet,\partial_S)$$

over the germ $(S,0)$ of $(\mathcal{K}_S^\bullet,\partial)$, i.e. ∂_S induces ∂ on the special fibre. By **??** $H^p(\mathcal{K}_S^\bullet)=0$ for $p\neq 0$ and $\mathcal{F}:=H^0(\mathcal{K}_S^\bullet)$ is flat over S. Hence we obtain a natural functor

$$\mathbf{E}\longrightarrow\mathrm{Quot}_{\mathcal{K}^0}$$

of fibrations in groupoids. In particular we get induced maps of the infinitesimal deformations. Using the identification from **??**, **??** this amounts to a map

$$\beta:H^1\left(\mathrm{Hom}(\mathcal{K}^\bullet,\mathcal{K}^\bullet\otimes p_2^*\mathcal{M})\right)\longrightarrow\mathrm{Hom}_{X_S}(\mathcal{G},\mathcal{F}\otimes p_2^*\mathcal{M})$$

where $\mathcal{G}:=\ker(\mathcal{K}_S^0\to\mathcal{F})$. Using the explicit form of the correspondences in the proofs of **??** and **??** this map associates to a morphism of complexes $\varphi:\mathcal{K}^\bullet\to\mathcal{K}^\bullet\otimes p_2^*\mathcal{M}$ of degree 1 the map

$$\beta(\varphi):=\varphi_*:\mathcal{G}\cong H^{-1}(\mathcal{K}^\bullet)\to\mathcal{F}\otimes p_2^*\mathcal{M}\cong H^0(\mathcal{K}^\bullet\otimes p_2^*\mathcal{M}).$$

Later on we need the following criterion for when β is surjective.

LEMMA 5.5.3. *If moreover $\mathcal{K}^\bullet$ is a bounded complex and if*

$$(V)\qquad \mathrm{Ext}^{p+1}_{X_S}\left(\mathcal{K}_S^k,\mathcal{K}_S^{k-p}\otimes p_2^*\mathcal{M}\right)=0\quad \textit{for all}\ p\geq 0\ \textit{and}\ k\leq -1\ ,$$

then β is surjective.

Proof. Consider the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{\mathcal{K}}^{-2} \arrow[r, "\bar{\partial}"] & \tilde{\mathcal{K}}^{-1} \arrow[r, "\bar{\partial}"] & \tilde{\mathcal{K}}^{0} \arrow[r] & \mathcal{F} \otimes p_2^* \mathcal{M} \arrow[r] & 0 \\
 & \mathcal{K}_S^{-3} \arrow[r, "\partial"] \arrow[u, dotted, "\psi^{-2}"] & \mathcal{K}_S^{-2} \arrow[r, "\partial"] \arrow[u, dotted, "\psi^{-1}"'] & \mathcal{K}_S^{-1} \arrow[r] \arrow[u, dotted, "\psi^{0}"'] & \mathcal{G} \arrow[r] \arrow[u, "\psi"'] & 0
\end{tikzcd}$$

of solid arrows, where $\tilde{\mathcal{K}}^j = \mathcal{K}_S^j \otimes p_2^* \mathcal{M}$. We have to find homomorphism ψ^i as indicated by the dotted arrows such that the above diagram becomes commutative. Assume that $\psi, \psi^0, \ldots, \psi^k$ $(k < 1)$ are already constructed. The composition $\psi^k \partial$ defines a morphism

$$\psi^k \partial : \mathcal{K}_S^k \longrightarrow \operatorname{Im}\left(\tilde{\mathcal{K}}^{k+1} \xrightarrow{\bar{\partial}} \tilde{\mathcal{K}}^{k+2}\right) =: \mathcal{Z}^{k+2}.$$

Using the exact $\operatorname{Ext}_{X_S}(\mathcal{K}^k, -)$-sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{Z}^{k+1} \longrightarrow \tilde{\mathcal{K}}^{k+1} \longrightarrow \mathcal{Z}^{k+2} \longrightarrow 0$$

the map $\psi^k \partial$ can be lifted to a morphism $\psi^{k-1} : \mathcal{K}_S^k \to \tilde{\mathcal{K}}_S^{k+1}$ if

$$(*) \qquad \operatorname{Ext}^1_{X_S}\left(\mathcal{K}_S^k, \mathcal{Z}^{k+1}\right) = 0 \ .$$

The exact $\operatorname{Ext}_{X_S}(\mathcal{K}^k, -)$-sequences associated to the short exact sequences

$$0 \longrightarrow \mathcal{Z}^{k-l+1} \longrightarrow \tilde{\mathcal{K}}^{k-l+1} \longrightarrow \mathcal{Z}^{k-l+2} \longrightarrow 0$$

together with our assumption (V) show that

$$\operatorname{Ext}^l_{X_S}\left(\mathcal{K}_S^k, \mathcal{Z}^{k-l+2}\right) \cong \operatorname{Ext}^{l+1}_{X_S}\left(\mathcal{K}_S^k, \mathcal{Z}^{k-l+1}\right), \quad l \geq 1.$$

As by assumption $\mathcal{Z}^{k-l+1}$ vanishes for $l \gg 0$, $(*)$ follows. □

Applications to codimension 2 subspaces. A case of particular interest is when the complex in question arises as complex from a structure theorem. As a first example let

$$(*) \qquad 0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

be a locally free resolution of Y such that $Y \subseteq X$ is a subspace defined by an ideal $\mathcal{I} \subseteq \mathcal{O}_X$ of grade 2. For instance, the grade condition is automatically satisfied if X is Cohen-Macaulay and Y has codimension 2 in every point. We note the following simple lemma.

Lemma 5.5.4. *Let* $(S, 0)$ *be a germ of a complex space and* $\beta : \mathcal{E}_S \to \mathcal{F}_S$ *be a homomorphism with* $\beta(0) = \alpha$, *where the index* S *denotes the pullback to* $X \times S$. *Then there is an exact sequence*

$$(\sharp) \qquad 0 \longrightarrow \mathcal{E}_S \xrightarrow{\beta} \mathcal{F}_S \longrightarrow \mathcal{O}_{X \times S} \longrightarrow \mathcal{O}_{\mathcal{Y}} \longrightarrow 0$$

inducing $(*)$ *on the special fibre. Moreover, up to isomorphism this sequence is uniquely determined by* β.

Proof. The map β defines a map $\bigwedge^e \beta$, where $e := \operatorname{rk} \mathcal{E}$. We have canonical isomorphism

$$\det \mathcal{E}_S \cong \det \mathcal{F}_S \ , \quad \mathcal{F}_S^\vee \cong \bigwedge^e \mathcal{F}_S \otimes \det \mathcal{F}_S^\vee \ .$$

Therefore $\bigwedge^{e} \beta$ defines a section in

$$\bigwedge^{e} \mathcal{F}_S \otimes \det \mathcal{E}_S^\vee \cong \bigwedge^{e} \mathcal{F}_S \otimes \det \mathcal{F}_S^\vee \cong \mathcal{F}_S^\vee$$

so that $\bigwedge^{e} \beta$ amounts to a map $\gamma : \mathcal{F}_S \longrightarrow \mathcal{O}_{X \times S}$. The reader may verify that

$$(**) \qquad 0 \longrightarrow \mathcal{E}_S \xrightarrow{\beta} \mathcal{F}_S \xrightarrow{\gamma} \mathcal{O}_{X \times S} \longrightarrow \mathcal{O}_{\mathcal{Y}} := \operatorname{coker} \gamma \longrightarrow 0$$

is a complex which on the special fibre can be identified with $(*)$. Using **??** it follows that $(**)$ is exact. Conversely, applying $\mathcal{H}om_{\mathcal{O}_{X_S}}(-, \mathcal{O}_{X_S})$, it follows that the section $\gamma^\vee$ of $\mathcal{F}_S^\vee$ is the kernel of $\beta^\vee$, and is thus unique up to isomorphism. □

To apply the result above, consider a subspace $S \subseteq \operatorname{Hom}(\mathcal{E}, \mathcal{F})$ of finite dimension containing α and let $\beta : \mathcal{E}_S \to \mathcal{F}_S$ be the "tautological" $\mathcal{O}_{X_S}$–linear map with $\beta_S(s) = s$. We can consider the exact sequence $(**)$ constructed in **??** so that $\mathcal{Y} \subseteq X \times S$ is a closed subspace that is S–flat over a neighbourhood of $\alpha \in S$. We get an associated Kodaira-Spencer map

$$\text{(A)} \qquad \delta_{\mathrm{KS}} : S \to \operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \ .$$

By the considerations above this is the composite of the homomorphism

$$S \subseteq \operatorname{Hom}(\mathcal{E}, \mathcal{F}) \longrightarrow H^1(\mathcal{H}om(\mathcal{K}^\bullet, \mathcal{K}^\bullet)) \longrightarrow \operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$$

which we studied above; here $\mathcal{K}^\bullet$ is the complex $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{O}_X \to 0$. Thus we get the following results:

Corollary 5.5.5. *Assume that $Y \subseteq X$ is a compact subspace and that the groups $H^1(X, \mathcal{E}nd\,\mathcal{E})$, $H^1(X, \mathcal{E}nd\,\mathcal{F})$ and $H^2(X, \mathcal{H}om(\mathcal{F}, \mathcal{E}))$ vanish. Then we can choose S such that (A) is surjective. Moreover, the subspace $\mathcal{Y} \subseteq X \times S$ constructed in* **??** *is the versal embedded deformation of $Y \subseteq X$ (at the point $\alpha \in S$). In particular, the Douady space H_X is smooth at $[Y]$.*

The *proof* follows from the fact that the Kodaira-Spencer map

$$V \longrightarrow \operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$$

is surjective by **??** and **??**.

Note that if in addition the map $\operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \to \operatorname{Ex}(Y, \mathcal{O}_Y)$ is surjective, the deformation $\mathcal{Y}$ above is also a versal deformation of the compact complex space Y. Thus the following result holds.

Corollary 5.5.6. *Assume that the following conditions are satisfied.*

1. $H^1(X, \mathcal{E}nd\,\mathcal{E}) = H^1(X, \mathcal{E}nd\,\mathcal{F}) = H^2(X, \mathcal{H}om(\mathcal{F}, \mathcal{E})) = 0.$
2. $\operatorname{Ex}(X, \mathcal{O}_Y) = 0.$

Then the composed map

$$\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \longrightarrow \operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \longrightarrow \operatorname{Ex}(Y, \mathcal{O}_Y)$$

is surjective. Moreover, if the group $\operatorname{Ex}(Y, \mathcal{O}_Y)$ *has finite dimension and S above is chosen in such a way that it surjects onto* $\operatorname{Ex}(Y, \mathcal{O}_Y)$ *then the deformation $\mathcal{Y}$ of Y constructed above is formally versal for Y. In particular, the basis of a formally versal deformation is smooth.*

Examples 5.5.7. (1) Assume that $X = \mathbb{P}^n$ and that

$$\mathcal{E} = \bigoplus_{i=1}^{e} \mathcal{O}(-b_i) \quad \text{and} \quad \mathcal{F} = \bigoplus_{i=1}^{e+1} \mathcal{O}(-a_i)$$

are direct sums of line bundles on $\mathbb{P}^n$. As before let $\alpha : \mathcal{E} \to \mathcal{F}$ be an $\mathcal{O}_{\mathbb{P}^n}$-linear map; note that such a map is given by an $e \times (e+1)$ matrix of homogeneous polynomials. Assume as before that the subspace Y given by the vanishing of the maximal minors of α has codimension 2. If $n \geq 3$ then the vanishing conditions in the above result are automatically satisfied. Thus the versal embedded deformation of $Y \subseteq \mathbb{P}^n$ can be obtained by deforming the entries of the matrix α. If moreover $H^1(Y, \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_Y) = 0$ then even all deformations of Y are given in this way. Note that this last vanishing condition is automatically satisfied in the case that $n \geq 5$. This follows easily from the sequence

$$\mathcal{O} \longrightarrow \Theta_{\mathbb{P}^n} \otimes \mathcal{E} \longrightarrow \Theta_{\mathbb{P}^n} \otimes \mathcal{F} \longrightarrow \Theta_{\mathbb{P}^n} \longrightarrow \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_Y \longrightarrow 0$$

and the fact that $H^i(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(j))$ vanishes for $0 < i < n-1$ and all j.

Observe also that in the case $n = 4$ the group $H^1\left(\mathbb{P}^4, \Theta_{\mathbb{P}^4} \otimes \mathcal{O}_Y\right)$ does not vanish necessarily. For instance if $Y \subseteq \mathbb{P}^4$ is the intersection of two quadrics then Y is a $K3$–surface which admits always deformations that are not embeddable into $\mathbb{P}^4$.

(2) As a particular case consider the map

$$\mathcal{O}_{\mathbb{P}^3}(-5) \oplus \mathcal{O}_{\mathbb{P}^3}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-4)$$

given by the matrix

$$\begin{pmatrix} f & g & X_0 \\ X_1 & X_2 & 0 \end{pmatrix}$$

where $f, g \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ are sufficiently generic. The reader may easily verify that the curve Y given by the vanishing of the 2–minors of this matrix is the union of the line $\{X_1 = X_2 = 0\}$ and the plane curve $\{X_0 = fX_2 - gX_1 = 0\}$. Thus it has a unique singular point at $[0 : 0 : 0 : 1]$. Deforming the matrix the zero in the lower right corner always survives and so gives rise to a singular point on the deformed curve. It follows that Y cannot be deformed into a smooth curve in $\mathbb{P}^3$. Moreover, applying $\mathrm{Hom}(-, \omega_{\mathbb{P}^3})$ to the resolution $(\sharp)$ we obtain a presentation

$$\omega_{\mathbb{P}^3}(2)^{\oplus 2} \oplus \omega_{\mathbb{P}^3}(4) \longrightarrow \omega_{\mathbb{P}^3}(5) \oplus \omega_{\mathbb{P}^3}(3) \longrightarrow \omega_Y \longrightarrow 0$$

Hence the dualizing module of the affine cone over Y has no generator in degree 0. From **??** we deduce that all deformations of Y are embedded. Thus Y is not smoothable as a compact complex space. For further examples, see [**?**], [**Ste**].

(3) ([**?**]) Let X be a smooth surface and $Y \subseteq X$ a subscheme of dimension 0. Replacing X by a small Stein neighbourhood of Y in X we may assume that X is a Stein manifold and that Y admits a resolution

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

with free modules $\mathcal{E}$, $\mathcal{F}$ on X. Obviously, the conditions in **??** are satisfied. Hence we obtain that all deformations of Y in X are obtained by deforming α. In particular we obtain that the *Douady space* H_X *is smooth* at $[Y]$.

Exercise 5.5.8.

1. Given a (2×3)–matrix as in Example (2) above, assume that f, g are generic, but this time with $\deg f = \deg g = m \geq 1$. As before let Y denote the vanishing locus of the (2×2)–minors.

   (a) Determine the minimal resolution of $\mathcal{O}_Y$.
   (b) Discuss the (embedded) deformations of Y: are there any cases in which Y is smoothable as a curve in $\mathbb{P}^3$?
   (c) Are there any cases in which not all deformations of Y (as a complex space) are embeddable?

2. (see [**?**]) Assume that the complex

$$\mathcal{K}^\bullet : 0 \longrightarrow \mathcal{E} := \bigoplus_{i=1}^{e} \mathcal{O}(-b_i) \longrightarrow \mathcal{F} := \bigoplus_{i=1}^{e+1} \mathcal{O}(-a_i) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

on $\mathbb{P}^n$, $n \geq 2$, defines a subspace Y of codimension 2 in $\mathbb{P}^n$.

   (a) Using the method of proof in **??** show that

$$\begin{array}{rcl} H^1\left(\mathrm{Hom}(\mathcal{K}^\bullet, \mathcal{K}^\bullet)\right) & \longrightarrow & \mathrm{Hom}(\mathcal{I}_Y, \mathcal{O}_Y) \\ H^0\left(\mathrm{Hom}(\mathcal{K}^\bullet, \mathcal{K}^\bullet)\right) & \longrightarrow & \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) = \mathbb{C} \end{array}$$

   are bijective.

   (b) Show that $H^i\left(\mathrm{Hom}(\mathcal{K}^\bullet, \mathcal{O}_{\mathbb{P}^n})\right) = 0$ for $i > 0$ and that the map

$$H^2\left(\mathrm{Hom}(\mathcal{K}^\bullet, \mathcal{O}_{\mathbb{P}^n})\right) \longrightarrow H^2\left(\mathrm{Hom}(\mathcal{K}^\bullet, \mathcal{K}^\bullet)\right)$$

   induced by the inclusion $\mathcal{O}_{\mathbb{P}^n} \hookrightarrow \mathcal{K}^\bullet$ is bijective.

   (c) Deduce from a) and b) that the dimension of $H_{\mathbb{P}^n}$ at $[Y]$ is given by

$$\chi\left(\mathrm{Hom}(\mathcal{K}^\bullet, \mathcal{O}_{\mathbb{P}^n})\right) - \chi\left(\mathrm{Hom}(\mathcal{K}^\bullet, \mathcal{K}^\bullet)\right) - 1\ ,$$

   where for a finite complex $V^\bullet$ of finite dimensional $\mathbb{C}$–vector spaces $\chi(V^\bullet)$ denotes the Euler characteristic, i.e. $\chi(V^\bullet) = \sum (-1)^i \dim_{\mathbb{C}} V^i$. Show that this number is equal to

$$\sum_{b_i \geq a_j} \binom{b_i - a_j + n}{n} + \sum_{a_j \geq b_i} \binom{a_j - b_i + n}{n} - \sum_{b_i \geq b_j} \binom{b_i - b_j + n}{n} - \sum_{a_i \geq a_j} \binom{a_i - a_j + n}{n}.$$

3. Assume that the curve $Y \subseteq \mathbb{P}^3$ is given by a resolution as in Ex. 2 above.

   (a) Show that the degree $\deg Y$ and the arithmetic genus $p_a(Y) = 1 - \chi(\mathcal{O}_Y)$ are given by the formulas

$$\begin{array}{rcl} \deg Y & = & 1 - \sum_i \binom{a_i - 1}{2} + \sum \binom{b_i - 1}{2} \\ p_a(Y) & = & \sum_i \binom{b_i - 1}{3} - \sum \binom{a_i - 1}{3}\ . \end{array}$$

   (Hint: to compute $\deg Y$ show that $\chi\left(\mathcal{O}_{Y \cap H}\right)$ is given by the formula above, where $Y \cap H$ is a generic hypereplane section of Y.)

   (b) Show that a general map $\mathcal{E} = \mathcal{O}(-6)^{\oplus 2} \to \mathcal{F} = \mathcal{O}(-5)^{\oplus 2} \oplus \mathcal{O}(-2)$ defines a smooth curve $Y \subseteq \mathbb{P}^3$ of genus 12 and degree 9 and that the dimension of $H_{\mathbb{P}^3}$ at $[Y]$ is 38. But there is another component of $H_{\mathbb{P}^3}$ which parameterizes curves of degree 9 and arithmetic genus 12. It corresponds to maps $\mathcal{E} = \mathcal{O}(-4)^{\oplus 2} \oplus \mathcal{O}(-7) \to \mathcal{F} = \mathcal{O}(-3)^{\oplus 3} \oplus \mathcal{O}(-6)$. Show that each curve of this type is reducible and cannot be smoothed. Show that the dimension of this component of $H_{\mathbb{P}^3}$ is 39.

4. Show that the maximal minors of the $n \times (n+1)$ matrix

$$\begin{pmatrix} x & y & 0 & \cdots & 0 & 0 \\ 0 & x & y & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \end{pmatrix}$$

define the n^{th} power of the maximal ideal of $\mathcal{O}_{\mathbb{C}^2,0}$, thus the $(n-1)^{\text{st}}$ infinitesimal neighbourhood Y of 0 in $\mathbb{C}^2$. Determine a finite-dimensional vector space S of $n \times (n+1)$ matrices that will describe a semi-universal deformation of Y.

5.6. The case of codimension 3 Gorenstein subspaces

Let us now turn to the case that Y is a subspace of a given complex space X which admits a resolution

$$(*) \qquad 0 \longrightarrow \mathcal{L} \xrightarrow{\varphi^\vee \otimes \mathcal{L}} \mathcal{E}^\vee \otimes \mathcal{L} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

where α is given by a skew symmetric form in $H^0(X, \bigwedge^2 \mathcal{E} \otimes \mathcal{L}^\vee)$. Moreover we assume that we always have the following conditions

$$(**) \qquad \operatorname{grade}_Y \mathcal{I} \geq 3 \quad \text{where} \quad \mathcal{I} \subseteq \mathcal{O}_X \text{ is the ideal sheaf of } Y \ .$$

Note that the above conditions ensure that

$$\operatorname{rk} \mathcal{E} = 2r + 1 \text{ is odd.}$$

Furthermore taking determinants in $(*)$ gives that

$$\mathcal{L}^r := \mathcal{L}^{\otimes r} \cong \det \mathcal{E} \ .$$

It is well know that one can reconstruct in a natural way the sequence $(*)$ from the knowledge of α. This is done as follows. Consider α as an element in $\operatorname{Hom}(\bigwedge^2 \mathcal{E}^\vee, \mathcal{L}^\vee)$. Taking the r^{th} power $\alpha^{(r)}$ defines a map $\bigwedge^{2r} \mathcal{E}^\vee \to \mathcal{L}^{-r}$, i.e. a section in $\bigwedge^{2r} \mathcal{E} \otimes \mathcal{L}^{-r}$. As $\mathcal{L}^{-r} = \det \mathcal{E}$ this amounts to a section $\bigwedge^r \alpha \in H^0(X, \mathcal{E}^\vee)$. Thus we get maps

$$\mathcal{L} \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \ , \quad \mathcal{E} \longrightarrow \mathcal{O}_X \ ,$$

and it is easy to verify that

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \xrightarrow{\alpha} \mathcal{E} \longrightarrow \mathcal{O}_X$$

is a complex. Moreover, in points, where α has maximal rank, this sequence is exact and is just the sequence $(*)$ up to isomorphism.

We note the following lemma.

Lemma 5.6.1. *Under the assumptions above, let $(S, 0)$ be a complex space germ and $\beta \in H^0(X \times S, \bigwedge^2 \mathcal{E}_S \otimes \mathcal{L}_S)$ a skew symmetric map inducing α on the special fibre. Then there is an exact sequence*

$$(\sharp) \qquad 0 \longrightarrow \mathcal{L}_S \longrightarrow \mathcal{E}_S^\vee \otimes \mathcal{L}_S \xrightarrow{\beta} \mathcal{E}_S \longrightarrow \mathcal{O}_{X \times S} \longrightarrow \mathcal{O}_{\mathcal{Y}} \longrightarrow 0$$

inducing $(*)$ *over 0. Moreover,* $\mathcal{Y} \subseteq X \times S$ *is a closed subspace which is flat over* S.

Proof. By the construction preceding the lemma we get a complex $(\sharp)$ inducing $(*)$ over 0. By **??** it follows that $(\sharp)$ is exact and $\mathcal{Y}$ is flat over S. $\square$

Let us now consider deformations of a given skew symmetric map $\alpha \in H^0(\overset{2}{\bigwedge}\mathcal{E}\otimes \mathcal{L}^\vee)$: such a deformation open a germ $(S,0)$ consists of a skew symmetric map β as in the preceding lemma. It defines in particular a subspace $\mathcal{Y} \subseteq X \times S$ which is S–flat. Thus we get a functor from the category of deformations of α into the deformations of Y.

In a next step we want to compute the associated map of infinitesimal deformations. Let $\beta \in H^0(X \times S, \overset{2}{\bigwedge}\mathcal{E}_S \otimes \mathcal{L}_S^\vee)$ be given and assume that $\mathcal{M}$ is a coherent $\mathcal{O}_S$–module. A skew symmetric map

$$\tilde{\beta} : \mathcal{E}^\vee_{S[\mathcal{M}]} \otimes \mathcal{L}_{S[\mathcal{M}]} \longrightarrow \mathcal{E}_{S[\mathcal{M}]}$$

has the form $\tilde{\beta} = \beta + \alpha'\varepsilon$ with a section α' in $(\overset{2}{\bigwedge}\mathcal{E}_S^\vee) \otimes \mathcal{L}_S \otimes p_S^*\mathcal{M}$. Thus the space of infinitesimal deformation for skew symmetric maps is

$$\operatorname{Ex}(\beta/S, \mathcal{M}) \cong H^0(X \times S, \overset{2}{\bigwedge}\mathcal{E}_S^\vee \otimes \mathcal{L}_S \otimes p_S^*\mathcal{M}) \ .$$

The above functor defines a map of infinitesimal deformations

$$\vartheta : H^0(X \times S, \overset{2}{\bigwedge}\mathcal{E}_S^\vee \otimes \mathcal{L}_S \otimes p_S^*\mathcal{M}) \longrightarrow \operatorname{Hom}_{X\times S}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_\mathcal{Y})$$

where $\mathcal{J} \subseteq \mathcal{O}_{X\times S}$ is the ideal sheaf of $\mathcal{Y}$.

We wish to give an explicit description of this map in terms of linear algebra.

Given $\tilde{\beta}$ we can define as above

$$\tilde{\beta}^{(r)} \in \operatorname{Hom}\left(\mathcal{E}^\vee_{S[\mathcal{M}]}, \mathcal{L}^\vee_{S[\mathcal{M}]}\right) \ .$$

Module $\mathcal{M}$ this induces $\beta^{(r)}$, and since $\varepsilon^2 = 0$, we have

$$\tilde{\beta}^{(r)} = \beta^{(r)} + \beta^{(r-1)}\alpha'\varepsilon \ ,$$

see **??**. Multiplying by $\tilde{\beta}$ gives $\tilde{\beta}^{(r)}\tilde{\beta} = 0$, and so

$$\begin{aligned} 0 &= \left(\beta^{(r)} + \beta^{(r-1)}\alpha'\varepsilon\right)(\beta + \alpha'\varepsilon) \\ &= \varepsilon\left(\beta^{(r-1)}\alpha'\beta + \beta^{(r)}\alpha'\right) \ , \end{aligned}$$

i.e. the diagram

$$\begin{tikzcd}
\mathcal{E}_S^\vee \otimes \mathcal{L}_S \arrow[r, "\beta"] \arrow[d, "\alpha'"'] & \mathcal{E}_S \arrow[r, "\beta^{(r)}"] \arrow[d, "\beta^{(r-1)}\alpha'"] & \mathcal{J} \arrow[d, dotted] \\
\mathcal{E}_S \otimes p_S^*\mathcal{M} \arrow[r, "-\beta^{(r)}"] & p_S^*\mathcal{M} \arrow[r] & \mathcal{O}_\mathcal{Y} \otimes p_S^*\mathcal{M}
\end{tikzcd}$$

commutes. In particular it defines a map as indicated by the dotted arrow.

We claim:

Lemma 5.6.2. $\vartheta(\alpha')$ *is the map* $\mathcal{J} \longrightarrow \mathcal{O}_\mathcal{Y} \otimes p_S^*\mathcal{M}$ *induced by* $\beta^{(r-1)}\alpha'$.

PROOF. This is an easy consequence of the fact that $\tilde{\beta}$ defines the subspace $\tilde{\mathcal{Y}}$ given by the complex

$$\mathcal{E}^{\vee}_{S[\mathcal{M}]} \otimes \mathcal{L}_{S[\mathcal{M}]} \xrightarrow{\tilde{\beta}} \mathcal{E}_{S[\mathcal{M}]} \xrightarrow{\tilde{\beta}^{(r)}} \mathcal{O}_{X_{S[\mathcal{M}]}} \longrightarrow \mathcal{O}_{\tilde{\mathcal{Y}}} \longrightarrow 0 \ .$$

□

In order to give criteria for when every deformation of Y comes from a deformation of the skew symmetric map we need the following crucial proposition.

PROPOSITION 5.6.3. *There is a resolution*

$$0 \to \mathcal{E}^{\vee} \otimes \mathcal{L} \xrightarrow{g_2} \big((S^2\mathcal{E})^{\vee} \otimes \mathcal{L}\big) \oplus \mathcal{E} \xrightarrow{g_1} \mathcal{E}^{\vee} \otimes \mathcal{E} \xrightarrow{g_0}$$
$$\mathcal{H}om(\bigwedge^2 \mathcal{E}^{\vee}, \mathcal{L}^{\vee}) \xrightarrow{\vartheta} \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathcal{Y}}) \to 0 \ ,$$

where ϑ is the map of infinitesimal $\mathrm{Hom}(\bigwedge^2 \mathcal{E}^{\vee}, \mathcal{L}^{\vee})$ *deformations defined above (for the case that S is a simple point and $\mathcal{M} = \mathcal{O}_S$).*

PROOF. Let us first define the maps g_i, $i = 0, 1, 2$. Let $g_2(e^* \otimes \lambda) := (e^* \vee \varphi^{\vee} \otimes \lambda, -\alpha(e^* \otimes \lambda))$

$$\begin{aligned} g_1\left((e_1^* \vee e_2^*) \otimes \lambda, e\right) &:= e_2^* \otimes \alpha(e_1^* \otimes \lambda) + e_1^* \otimes \alpha(e_2^* \otimes \lambda) + \varphi^{\vee} \otimes e \\ g_0(e^* \otimes e) &:= [\alpha(e^* \otimes \lambda) \otimes e - e \otimes \alpha(e^* \otimes \lambda)]\, \lambda \end{aligned}$$

where λ is a local basis of $\mathcal{L}$ and e, e^* are local sections in $\mathcal{E}^{\vee}$ resp. $\mathcal{E}$. (Note that $\mathrm{Hom}(\bigwedge^2 \mathcal{E}^{\vee}, \mathcal{L}^{\vee})$ is considered as a subsheaf of $\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{L}^{\vee}$.) Using the relations

$$\alpha(\varphi^{\vee} \otimes \lambda) = 0 \ , \quad \varphi \circ \alpha = 0$$

the reader may easily verify that $g_1 \circ g_2 = 0$ and $g_0 \circ g_1 = 0$. Recall that $\vartheta(\alpha')$ is the map induced by

$$\alpha^{(r-1)}\alpha' : \mathcal{E} \longrightarrow \mathcal{O}_X \ .$$

Assume that $\vartheta(\alpha') = 0$, i.e. $\alpha'\alpha^{(r-1)}(\mathcal{E}) \subseteq \mathcal{F}_Y$. Then locally we find a map β in $\mathcal{H}om(\mathcal{E}, \mathcal{E}) = \mathcal{E}^{\vee} \otimes \mathcal{E}$ such that $\alpha^{(r)} \cdot \beta = \alpha'\alpha^{(r-1)}$.[1] Hence $\alpha \cdot \beta = \alpha'$; observe that α has generally maximal rank r and so

$$\alpha^{(r-1)} : \bigwedge^2 \mathcal{E} \otimes \mathcal{L}^{\vee} \longrightarrow \bigwedge^{2r} \mathcal{E} \otimes \mathcal{L}^{-r} \cong \mathcal{H}om(\mathcal{E}^{\vee}, \mathcal{O}_X)$$

is injective. But the equation $\alpha \cdot \beta = \alpha'$ is equivalent to $g_0(\beta) = \alpha'$ (up to a sign ?) This also shows $\vartheta \circ g_0 = 0$.

It remains to show that the rest of the constructed sequence is exact. Because of **??** it is sufficient to show this at points in $X \setminus Y$, i.e. we may suppose that

a) $\mathcal{L} \cong \mathcal{O}_X$
b) $\mathcal{E} \cong \mathcal{F} \otimes \mathcal{O}_X$ and

$$\alpha : \mathcal{F}^{\vee} \otimes \mathcal{O}_X \longrightarrow \mathcal{F} \otimes \mathcal{O}_X$$

has the form $(\beta, 0)$ for some isomorphism β with $\beta^{\vee} = -\beta$.

But in this case the reader may easily verify that the above sequence is in fact exact. □

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[1] Note that inner multiplication gives: $\alpha^{(r)} \cdot \beta = \alpha^{(r-1)}(\alpha \cdot \beta)$

COROLLARY 5.6.4. *Assume that*

$$H^3(X,\mathcal{E}^\vee\otimes\mathcal{L})=H^2(X,\mathcal{E})=H^2(X,S^2\mathcal{E}^\vee\otimes\mathcal{L})=H^1(X,\mathcal{E}^\vee\otimes\mathcal{E})=0\ .$$

Then the map

$$\vartheta : H^0(X,\bigwedge^2\mathcal{E}\otimes\mathcal{L}^\vee)\longrightarrow \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_{\mathcal{Y}})$$

is surjective. In particular, if $S\subseteq H^0(X,\bigwedge^2\mathcal{E}\otimes\mathcal{L}^\vee)$ *is a finite dimensional subspace containing* α *and mapping surjectively onto* $\mathrm{Hom}(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_{\mathcal{Y}})$ *then the tantolocial sequence* ($\sharp$) *defines a subspace* $\mathcal{Y}\subseteq X\times S$ *which is flat over* (S,α)*, and which is formally versal for the embedded deformation* $Y\subseteq X$*. In particular, if moreover* $Y\subseteq X$ *is a compact subspace then the Douady space* H_X *is smooth in* $[Y]$.

COROLLARY 5.6.5. *Assume that the composite map*

$$H^0(X,\bigwedge^2\mathcal{E}\otimes\mathcal{E}^\vee)\longrightarrow \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_Y)\longrightarrow \mathrm{Ex}(Y,\mathbb{C})$$

is surjective and Y *is finite dimensional.* $S\subseteq H^0(X,\bigwedge^2\mathcal{E}\otimes\mathcal{L}^\vee)$ *be a linear subspace of finite dimension containing* α*, and let* $\mathcal{Y}\subseteq X\times S$ *be as above. Then* $\mathcal{Y}$ *is a formally versal deformation of* Y *over the base space* (S,α)*. In particular, the basis of a versal deformation of* Y *is smooth.*

EXAMPLES 5.6.6. (1) Let X be a threefold and $Y\subseteq X$ a compact subspace of dimension 0 which is Gorenstein. Then the Douady space H_X is smooth at $[Y]$.

(2) Let $Y\subseteq\mathbb{P}^n$ be an arithmetically Gorenstein subschemes of codimension 3. By the structure theorem of Buchsbaum-Eisenbud there is a resolution

$$0\longrightarrow\mathcal{L}\longrightarrow\mathcal{E}\otimes\mathcal{L}\xrightarrow{\alpha}\mathcal{E}\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_Y\longrightarrow 0$$

where α is skew symmetric and $\mathcal{E}$ is a direct sum of line bundles. If $n\geq 4$ then the conditions in **??** are satisfied. Thus all embedded deformation of $Y\subseteq X$ are obtained by deforming α as a skew symmetric map.

CHAPTER 6

Openness of versality and applications

6.1. Openness of versality

Let $p : \mathbf{F} \to \mathbf{An}_\Sigma$ be a deformation theory. In this section we will examine the question as to when for an object $a \in \mathbf{F}(S)$ over a complex space $S \in \mathbf{An}_\Sigma$ the set of points

$$\{s \in S \mid a \text{ is formally versal at } s\}$$

is an open subset of S. If this is the case then we also say in brief that *openness of formal versality holds for a.* The aim of this section is to give a fairly general criterion for openness of formal versality for any deformation theory.

A crucial condition in our criterion is the existence of a so called obstruction theory for a. Roughly speaking, an *obstruction theory* assigns to each extension $S \hookrightarrow S'$ of S by $\mathcal{M}$ an obstruction $\mathrm{ob}(S \hookrightarrow S')$ in a suitable module which vanishes if and only if there is an extension $a \hookrightarrow a'$ in $\mathbf{F}$ over $S \hookrightarrow S'$. For instance, in the case of a smooth proper holomorphic map $f : X \longrightarrow S$ with unobstructed fibres we will see that we can take the trivial obstruction theory which is identically zero. Thereby we will verify that the set of points $s \in S$ in which f is the versal deformation of its fibre, is Zariski open in S. More generally, in later sections we will show that for all the deformation theories considered so far, e.g. deformations of compact complex spaces, of modules, of subspaces and of singularities, one has natural obstruction theories so that our criteria are satisfied. Thus in these examples we always have openness of versality. In the case of deformations of modules and locally trivial deformations of compact complex spaces we will prove the existence of a natural obstruction theory in the next section. However, for the general case of deformations of compact complex spaces and singularities we need the machinery of cotangent complexes and so we postpone this construction to Chapt. 6.

Before proving a first criterion for openness of formal versality we need a few preparations. First we provide a simple criterion for when an open subset of a complex space is Zariski open.

Lemma 6.1.1. *Let S be a complex space and $V \subseteq S$ a subset. Then the following condition are equivalent.*

(1) *V is Zariski open.*
(2) *For every closed irreducible analytic subset $T \subseteq S$ meeting V there is a Zariski open dense subset $U \subseteq T$ such that $U \subseteq V$.*

Proof. The implication (1)$\Longrightarrow$(2) is obvious. For the converse, observe first that it is sufficient to show that $V \cap S'$ is Zariski open for every irreducible component S' of S. Hence replacing S by S' we may suppose that S is irreducible. If $V \neq \emptyset$ then by assumption there is a Zariski open dense subset $U \subseteq S$ such that $U \subseteq V$. Then $A := S \backslash U$ is a proper analytic subset of S. By induction we may suppose that $V \cap A$ is Zariski open in A or, equivalently, that $A \backslash (V \cap A)$ is an analytic subset of A. As $U \subseteq V$,

$$A \backslash (V \cap A) = S \backslash V$$

and so V is Zariski open. □

Let now $p : \mathbf{F} \longrightarrow \mathbf{An}_\Sigma$ be as above and consider an object $a \in F$ lying over $S \in \mathbf{An}_\Sigma$. In 3.3.1 we defined for a coherent $\mathcal{O}_S$-module $\mathcal{M}$ the groups

$$\mathrm{Ex}_\Sigma(a, \mathcal{M}), \qquad \mathrm{Ex}_\Sigma(a/S, \mathcal{M})$$

$$\mathrm{Aut}_\Sigma(a, \mathcal{M}), \qquad \mathrm{Aut}_\Sigma(a/S, \mathcal{M}).$$

We can sheafify these constructions to obtain sheaves on S which we denote by

$$\mathcal{E}x_\Sigma(a, \mathcal{M}), \qquad \mathcal{E}x_\Sigma(a/S, \mathcal{M})$$

$$\mathcal{A}ut_\Sigma(a, \mathcal{M}), \qquad \mathcal{A}ut_\Sigma(a/S, \mathcal{M}).$$

For instance, $\mathcal{E}x_\Sigma(a, \mathcal{M})$ is the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Ex}_\Sigma(a|U, \mathcal{M}|U), \quad U \subseteq S \text{ open}.$$

Similarly we can form the sheaf $\mathcal{E}x(S/\Sigma, \mathcal{M})$ derived from the groups $\mathrm{Ex}(S/\Sigma, \mathcal{M})$. With these notations we can state the following variants of 3.3.3,3.3.4 and 3.4.15.

Proposition 6.1.2. *(1) For every exact sequence*

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow O$$

there is an exact sequence of sheaves

$$\begin{aligned} 0 \to \mathcal{A}ut_\Sigma(a, \mathcal{M}') \to \mathcal{A}ut_\Sigma(a, \mathcal{M}) \to \mathcal{A}ut_\Sigma(a, \mathcal{M}'') \to \\ \mathcal{E}x_\Sigma(a, \mathcal{M}') \to \mathcal{E}x_\Sigma(a, \mathcal{M}) \to \mathcal{E}x_\Sigma(a, \mathcal{M}''), \end{aligned}$$

and similarly for

$$\mathcal{A}ut_\Sigma(a/S, -) \quad , \quad \mathcal{E}x_\Sigma(a/S, -).$$

(2) There is a Kodaira Spencer sequence

$$\begin{aligned} 0 \to \mathcal{A}ut_\Sigma(a/S, \mathcal{M}) \to \mathcal{A}ut_\Sigma(a, \mathcal{M}) \to \mathcal{D}er_\Sigma(\mathcal{O}_S, \mathcal{M}) \to \\ \mathcal{E}x_\Sigma(a/S, \mathcal{M}) \to \mathcal{E}x_\Sigma(a, \mathcal{M}) \to \mathcal{E}x_\Sigma(S/\Sigma, \mathcal{M}). \end{aligned}$$

(3) a is formally versal at $s \in S$ if and only if $\mathcal{E}x_\Sigma(a, \mathcal{O}_S/\mathfrak{m}_s)_s = 0$.

Proof. (1) and (2) follow immediately from 3.3.3, 3.3.4 respectively. Moreover (3) is a reformulation of 3.4.15. □

We also note the following simple fact.

Lemma 6.1.3. *The sheaf $\mathcal{E}x(S/\Sigma, \mathcal{M})$ is coherent for every coherent $\mathcal{O}_S$-module $\mathcal{M}$.*

Proof. Since the question is local we many suppose that S admits a closed Σ-embedding into some open subset, say, M of $\mathbb{C}^n \times \Sigma$. Then it follow from the Zariski-Jacobi sequence that

$$\mathcal{E}x(S/\Sigma, \mathcal{M}) \cong \mathrm{Coker}(\mathcal{D}er_\Sigma(\mathcal{O}_M, \mathcal{M}) \to \mathcal{H}om_S(\mathcal{J}/\mathcal{J}^2, \mathcal{M})),$$

where $\mathcal{J} \subseteq \mathcal{O}_M$ is the ideal sheaf of $S \hookrightarrow M$. Hence $\mathcal{E}x(S/\Sigma, \mathcal{M})$ is coherent. □

For the formulation of the results below it is useful to introduce the following notation.

Definition 6.1.4. We will say that an additive functor $A : \mathbf{Coh}(S) \to \mathbf{Coh}(S)$ satisfies the *generic principle* if the following condition is satisfied.

(GP) For every closed reduced subspace T of S there is a Zariski open dense subset $U \subseteq T$ such that the canonical map

$$A(\mathcal{O}_T) \otimes \mathcal{O}_T/\mathfrak{m}_t \xrightarrow{\sim} A(\mathcal{O}_T/\mathfrak{m}_t)$$

is bijective for all $t \in U$.

We are now able to prove the following first criterion for openness of versality.

Theorem 6.1.5. *Assume that for $a \in \mathbf{F}$ over $S \in \mathbf{An}_\Sigma$ the following two conditions are satisfied.*

(1) *$\mathcal{E}x_\Sigma(a, \mathcal{M})$ is a coherent $\mathcal{O}_S$-module for all $\mathcal{M} \in \mathbf{Coh}(S)$.*
(2) *The functor $\mathcal{E}x_\Sigma(a, -)$ satisfies the generic principle.*

Then openness of formal versality holds for a.

Proof. We need to show that the set

$$V := \{s \in S : \mathcal{E}x_\Sigma(a, \mathcal{O}_S/\mathfrak{m}_s) = 0\}$$

is Zariski open in S. For this we verify condition (2) of our criterion given in 6.1.1. Let $T \subseteq S$ be a closed reduced subspace meeting V. By assumption there is a Zariski open dense subset $U_1 \subseteq T$ such that

$$\mathcal{E}x_\Sigma(a, \mathcal{O}_T) \otimes \mathcal{O}_T/\mathfrak{m}_t \cong \mathcal{E}x_\Sigma(a, \mathcal{O}_T/\mathfrak{m}_t). \tag{3}$$

Consider the Zariski open subset of T

$$U := U_1 \cap (T \backslash \operatorname{supp}(\mathcal{E}x_\Sigma(a, \mathcal{O}_T))).$$

We claim that
(i) $U \neq \emptyset$, and
(ii) $U \subseteq V$.
In view of 6.1.1 this will prove the result. (ii) is immediate as $U \subseteq U_1$ and so (3) holds. In order to show (i) consider a point $t \in T \cap V$, so that $\mathcal{E}x_\Sigma(a, \mathcal{O}_T/\mathfrak{m}_t) = 0$. By a simple induction on n it follows that $\mathcal{E}x_\Sigma(a, \mathcal{O}_T/\mathfrak{m}_t^n) = 0$ for all $n \geq 0$. By **??** the canonical map

$$\mathcal{E}x_\Sigma(a, \mathcal{O}_T)_t^\wedge \longrightarrow \varprojlim \mathcal{E}x_\Sigma(a, \mathcal{O}_T/\mathfrak{m}_t^n)$$

is injective, where $^\wedge$ denotes the $\mathfrak{m}_t$-adic completion. Hence we deduce that the stalk $\mathcal{E}x_\Sigma(a, \mathcal{O}_T)_t$ vanishes and so $T \backslash \operatorname{supp}(\mathrm{Ex}_\Sigma(a, \mathcal{O}_T))$ is non-empty. It follows that U is also nonempty as required. □

Usually the modules $\mathcal{E}x_\Sigma(a, \mathcal{M})$ are difficult to describe, so that in practise it is hard to verify the assumptions of 6.1.5. Therefore we will reformulate the criterion above into conditions for the sheaves $\mathcal{E}x_\Sigma(a/S, \mathcal{M})$ and certain modules which we call obstruction modules. We will introduce them as follows.

Definition 6.1.6. An *obstruction theory for $a \in \mathbf{F}(S), S \in \mathbf{An}_\Sigma$*, consists in a functor

$$\mathcal{O}b(a, -) : \mathbf{Coh}(S) \longrightarrow \mathbf{Coh}(S),$$

such that the following condition is satisfied.

(Ob) For every $\mathcal{M} \in \mathbf{Coh}(S)$ there is a map

$$\mathrm{ob} : \mathcal{E}x(S/\Sigma, \mathcal{M}) \longrightarrow \mathcal{O}b(a, \mathcal{M})$$

which is functorial in $\mathcal{M}$ so that the sequence

$$\mathcal{E}x_\Sigma(a, \mathcal{M}) \longrightarrow \mathcal{E}x(S/\Sigma, \mathcal{M}) \longrightarrow \mathcal{O}b(a, \mathcal{M})$$

is exact.

In other words, given a Σ-extension $S \hookrightarrow S'$ of S by $\mathcal{M} \in \mathbf{Coh}(S)$ then $\mathrm{ob}([S']) = 0$ in $\mathcal{O}b(a, \mathcal{M})$ if and only if locally in S we can find an extension $a \hookrightarrow a'$ of a by $\mathcal{M}$ over $S \hookrightarrow S'$.

Examples 6.1.7. A particularly simple case is when for every point $s \in S$ the object $a(s) := a \otimes \mathcal{O}_S/\mathfrak{m}_S$ admits a versal deformation, say, b over a germ (T, s) (for simplicity we suppress the dependence on s), such that T is smooth over Σ. We claim that in this case we can take $\mathcal{O}b(a, \mathcal{M}) := 0$ as an obstruction theory. In fact, if locally around the point s there is an extension $S \hookrightarrow S'$ of S by $\mathcal{M}$ then we can find an extension $a \hookrightarrow a'$ over $S \hookrightarrow S'$ as follows. By versality, there is a morphism of germs $f : (a, s) \to (b, s)$. Let $g := p(f) : (S, s) \longrightarrow (T, s)$ be the underlying morphism of complex spaces. Since T is smooth over Σ, g can be lifted to a morphism $g' : (S', s) \to (T, s)$. Then $a \hookrightarrow a' := a \times_S S'$ is the desired extension of a.

We are now able to rephrase 6.1.5 in terms of an obstruction theory as follows.

Theorem 6.1.8. *Assume that $a \in \mathbf{F}(S)$ admits an obstruction theory such that the following conditions are satisfied.*

(O1) *For every every coherent $\mathcal{O}_S$-module $\mathcal{M}$ the sheaf $\mathcal{E}x(a/S, \mathcal{M})$ is coherent.*
(O2) *The functors $\mathcal{E}x_\Sigma(a/S, -)$ and $Ob(a, -)$ on $\mathbf{Coh}(S)$ satisfy the generic principle (see 6.1.4).*

Then openness of formal versality holds for a.

Proof. We will verify conditions (1) and (2) of 6.1.5. The first one follows from the extended Kodaira Spencer sequence

$$\mathcal{D}er_\Sigma(\mathcal{O}_S, \mathcal{M}) \to \mathcal{E}x_\Sigma(a/S, \mathcal{M}) \to \mathcal{E}x_\Sigma(a, \mathcal{M}) \to \mathcal{E}x(S/\Sigma, \mathcal{M}) \to \mathcal{O}b(a, \mathcal{M}),$$

since the four outer terms are coherent by (O1) and 6.1.3. Hence $\mathcal{E}x_\Sigma(a, \mathcal{M})$ is coherent too.

By part (b) of the following lemma, the functors $\mathcal{D}er_\Sigma(\mathcal{O}_S, -)$ and $\mathcal{E}x(S/\Sigma, -)$ satisfy the generic principle. Hence the second condition also follows from the Kodaira-Spencer sequence using part (a) of the following lemma. ☐

In the proof above we have used the following simple observation.

Lemma 6.1.9. *(a) Let A, B, C, D, $E : \mathbf{Coh}(S) \to \mathbf{Coh}(S)$ be additive functors such that for every $\mathcal{M} \in \mathbf{Coh}(S)$ there is a functorial exact sequence*

$$A(\mathcal{M}) \to B(\mathcal{M}) \to C(\mathcal{M}) \to D(\mathcal{M}) \to E(\mathcal{M}).$$

Assume that A, B, D, E satisfy the generic principle. Then C also satisfies the generic principle.

(b) The functors $\mathcal{D}er_\Sigma(\mathcal{O}_S, -)$ and $\mathcal{E}x(S/\Sigma, -)$ satisfy the generic principle.

Proof. Let $T \subseteq S$ be a closed reduced subspace. In order to prove (a), choose a Zariski open dense subset U_1 of T such that, for F any one of the functors A, B, D, E, the map

$$F(\mathcal{O}_T)(t) := F(\mathcal{O}_T) \otimes \mathcal{O}_T/\mathfrak{m}_t \longrightarrow F(\mathcal{O}_T/\mathfrak{m}_t)$$

is bijective for all $t \in U_1$. Moreover choose a Zariski open dense subset U_2 of T such that the modules in the exact sequence

$$(*) \qquad A(\mathcal{O}_T) \to B(\mathcal{O}_T) \to C(\mathcal{O}_T) \to D(\mathcal{O}_T) \to E(\mathcal{O}_T) \to E(\mathcal{O}_T)/D(\mathcal{O}_T) \to 0$$

are all locally free on U_2. Set $U := U_1 \cap U_2$ and consider for $t \in U$ the following diagram.

$$\begin{tikzcd}
A(\mathcal{O}_T)(t) \arrow[r] \arrow[d, "\alpha"', "\cong"] & B(\mathcal{O}_T)(t) \arrow[r] \arrow[d, "\beta"', "\cong"] & C(\mathcal{O}_T)(t) \arrow[r] \arrow[d, "\gamma"'] & D(\mathcal{O}_T)(t) \arrow[r] \arrow[d, "\delta"', "\cong"] & E(\mathcal{O}_T)(t) \arrow[d, "\varepsilon"', "\cong"] \\
A(\mathcal{O}_T/\mathfrak{m}_t) \arrow[r] & B(\mathcal{O}_T/\mathfrak{m}_t) \arrow[r] & C(\mathcal{O}_T/\mathfrak{m}_t) \arrow[r] & D(\mathcal{O}_T/\mathfrak{m}_t) \arrow[r] & E(\mathcal{O}_T/\mathfrak{m}_t).
\end{tikzcd}$$

The maps $\alpha, \beta, \delta, \varepsilon$ are bijective for $t \in U$ by the choice of U. The second line is exact by assumption. Moreover the first line is obtained from the exact sequence $(*)$ by tensoring with $\mathcal{O}_T/\mathfrak{m}_t$ and so is also exact. Now the 5-lemma gives that γ is an isomorphism, as required.

In order to show (b), let us first treat the functor $\mathcal{D}er_\Sigma(\mathcal{O}_S, -)$. We choose a Zariski open dense subset U such that $\Omega^1_{S/\Sigma} \otimes \mathcal{O}_T$ is locally free on U. Since $\mathcal{D}er_\Sigma(\mathcal{O}_S, \mathcal{M}) \cong \mathcal{H}om_S(\Omega^1_{S/\Sigma}, \mathcal{M})$ the canonical maps

$$\mathcal{D}er_\Sigma(\mathcal{O}_S, \mathcal{O}_T) \otimes \mathcal{O}_T/\mathfrak{m}_t \longrightarrow \mathcal{D}er_\Sigma(\mathcal{O}_S, \mathcal{O}_T/\mathfrak{m}_t)$$

are isomorphisms for $t \in U$, whence $\mathcal{D}er_\Sigma(\mathcal{O}_S, -)$ satisfies the generic principle.

Finally let us show that the functor $\mathcal{E}x(S/\Sigma, -)$ satisfies the generic principle. If we embed S locally into an open subset $M \subseteq \mathbb{C}^n \times \Sigma$ with ideal sheaf, say, $\mathcal{J} \subseteq \mathcal{O}_{\mathbb{C}^n \times \Sigma}$ then by the Zariski-Jacobi sequence

$$\mathcal{E}x(S/\Sigma, \mathcal{M}) \cong \operatorname{Coker}(\mathcal{D}er_\Sigma(\mathcal{O}_M, \mathcal{M}) \longrightarrow \mathcal{H}om_S(\mathcal{J}/\mathcal{J}^2, \mathcal{M})).$$

Choose $U := \operatorname{Reg}(\mathcal{J}/\mathcal{J}^2 \otimes \mathcal{O}_T)$, i.e. U is the set of all points $t \in U$ such that the stalk of $\mathcal{J}/\mathcal{J}^2 \otimes \mathcal{O}_T$ at t is free as $\mathcal{O}_{T,t}$-module. Note that this does not depend on the choice of the local embedding since for two different embeddings the conormal modules differ at most by a free direct summand. Using the diagram

$$\begin{tikzcd}
\mathcal{D}er_\Sigma(\mathcal{O}_M, \mathcal{O}_T)(t) \arrow[r] \arrow[d, "\cong"] & \mathcal{H}om_T(\mathcal{J}/\mathcal{J}^2 \otimes \mathcal{O}_T, \mathcal{O}_T)(t) \arrow[r] \arrow[d, "\cong"] & \mathcal{E}x(S/\Sigma, \mathcal{O}_T)(t) \arrow[r] \arrow[d] & 0 \\
\mathcal{D}er_\Sigma(\mathcal{O}_M, \mathcal{O}_T/\mathfrak{m}_t) \arrow[r] & \mathcal{H}om_T(\mathcal{J}/\mathcal{J}^2 \otimes \mathcal{O}_T, \mathcal{O}_T/\mathfrak{m}_t) \arrow[r] & \mathcal{E}x(S/\Sigma, \mathcal{O}_T/\mathfrak{m}_t) \arrow[r] & 0
\end{tikzcd}$$

it follows that the vertical arrow on the right hand side is bijective for $t \in U$.

□

Example 6.1.10. Let us consider unobstructed deformations of compact complex manifolds. More precisely, let $p : \mathbf{F} \to \mathbf{An}$ be the category of deformations of compact complex spaces (see **??**) and consider a proper smooth map $f : X \to S$, such that every fibre $X_s := f^{-1}(s)$, $s \in S$, is unobstructed, i.e. it admits a smooth versal deformation. Then by 6.1.7 we can take $\mathcal{O}b(a, -) = 0$ as an obstruction theory, with $a := (f : X \to S)$. Moreover, by **??** $\mathrm{Ex}(a/S, \mathcal{M})$ is isomorphic to $H^1(X, \Theta_{X/S} \otimes f^*\mathcal{M})$ and so there is a natural isomorphism

$$\mathcal{E}x(a/S, \mathcal{M}) \cong R^1 f_*(\Theta_{X/S} \otimes f^*\mathcal{M}).$$

In particular, this sheaf is coherent and (O1) is satisfied. Applying **??** to the cohomology functors $F^i(-) = R^i f_*(\Theta_{X/S} \otimes f^*(-))$ it follows that also (O2) is satisfied. Hence openness of formal versality holds for a.

For instance, this observation applies to families for which $H^2(X_s, \Theta_{X_s}) = 0$ for all $s \in S$ (see **??**). Similarly, if $\omega_{X_s} \cong \mathcal{O}_{X_s}$ for all $s \in S$ then again all fibres X_s are unobstructed, see **??**.

Other examples, where one has a trivial obstruction theory, are unobstructed deformations of modules or embedded deformations (cf. Sect. 4., 4.?). The reader may work out that in such cases again openness of versality holds.

Appendix: Inverse systems and the generic principle

In this appendix we provide two lemmata. The first one is on inverse systems and was used in the previous section. The second one is a simple criterion for when the generic principle is satisfied for an additive functor as in **??**. We follow the exposition given in [**Fl**]; for applications of these techniques to the comparison theorem and semicontinuity theorem see also **??**.

A lemma on inverse systems. Let us consider a noetherian ring A and an additive functor on the category $\mathbf{Mod}_A^f$ of finite A–modules into itself

$$F : \mathbf{Mod}_A^f \longrightarrow \mathbf{Mod}_A^f$$

satisfying the following two conditions.

(a) F is half exact.

(b) F is A–linear, i.e. for all finite A-modules M, N the map

$$F_* : \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_A\left(F(M), F(N)\right)$$

is A–linear.

Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a finitely generated graded A–Algebra. We can extend F to finite graded R–modules $M = \bigoplus_{i \in \mathbb{Z}} M$ by setting

$$F(M) := \bigoplus_{i \in \mathbb{Z}} F(M_i)\ .$$

If $f \in R$ is a homogeneous element then multiplication by f gives a homogeneous map $M @ > f >> M$ and so an induced map

$$F(f) : F(M) \to F(M)\ .$$

Taking this as the multiplication by f on $F(M)$ the reader may easily verify that $F(M)$ becomes a graded R–module and so F extends to an R–linear functor on the category of finite graded R–modules into the category of graded R–modules. Clearly this extension is again half exact.

The following simple proposition is essential to understand the compatability of F with inverse limits.

PROPOSITION 6.1.11. *If M is a finite graded R–module then $F(M)$ is also a finite graded R–module.*

Proof. The ring R can be written as a graded quotient of a polynomial ring $A[T_1, \ldots, T_n]$ where $\deg T_i = w_i$ for some $w_i \in \mathbb{N}$. Clearly we may assume that $R = A[T_1, \ldots, T_n]$. We will proceed by induction on the number of indeterminates n. For $n = 0$ there is nothing to show. So assume that $n > 0$. First assume that T_n is not a zero divisor on M so that $0 \to M \xrightarrow{T_n} M \to M/T_nM \to 0$ is exact. Applying F we get an exact sequence

$$F(M) \xrightarrow{T_n} F(M) \longrightarrow F(M/T_nM) \ .$$

As M/T_nM can be considered as a finite module over $\overline{R} = A[T_1, \ldots, T_{n-1}]$ we get from the induction hypothesis that $F(M/T_nM)$ is a finite $\overline{R}$–modules, equivalently, finite over R. It follows that

$$F(M)/T_nF(M) \subseteq F(M/T_nM)$$

is again finite over R. Hence, using the lemma of Nakayama in the graded case (see **??** {Mat}) $F(M)$ is also finite over R.

In the general case, consider the ascending chain of submodules $\operatorname{Ann}_M\left(T_n^k\right)$, $k \geq 0$, of M. Since M is noetherian, we find $k \geq 0$ such that $\operatorname{Ann}_M\left(T_n^k\right) = \operatorname{Ann}_M\left(T_n^{k+r}\right)$ for all r. It follows that on $\overline{M} := M/\operatorname{Ann}_M(\left(T_n^k\right)$ the element T_n is not a zerodivisor and so by the first part of the proof $F(\overline{M})$ is finite over R. Using the exact sequence

$$F\left(\operatorname{Ann}_M(T_n^k)\right) \longrightarrow F(M) \longrightarrow F(\overline{M})$$

it remains to show that $F\left(\operatorname{Ann}_M(T_n^k)\right)$ is finite over R. But this again is a consequence of the induction hypothesis since $\operatorname{Ann}_M\left(T_n^k\right)$ is already finite over the subring $A[T_1, \ldots, T_{n-1}]$ of R. □

In the following we consider inverse systems of A–modules $\{F_n\}_{n\in\mathbb{N}}$ (see [**AMa**, Chapt. 10], for example). We call such a system *essentially zero* if for all n there is an $n' \geq n$ so that $F_{n'} \to F_n$ is the zero map. In particular then $\varprojlim F_n = 0$. Moreover, let

$$\{\psi_n\} : \{F_n\} \longrightarrow \{G_n\}$$

be a morphism of inverse systems, i.e. $F_n \to G_n$ is a collection of A–linear maps which are compatible with the given maps $F_{n+1} \to F_n$, $G_{n+1} \to G_n$. We call $\{\psi_n\}$ *essentially injective* if $\{\operatorname{Ker}\psi_n\}$ is essentially zero. The reader may immediately verify that if $\{\psi_n\}$ is essentially injective then the induced map

$$\varprojlim \psi_n : \varprojlim F_n \longrightarrow \varprojlim G_n$$

is injective.

Corollary 6.1.12. *Let N be a finite A–module and $\mathfrak{a} \subseteq A$ an ideal. Then the natural map*

$$\{F(M)/\mathfrak{a}^nF(M)\} \longrightarrow \{F(M/\mathfrak{a}^nM)\}$$

is essentially injective. In particular, the map $\varprojlim F(M)/\mathfrak{a}^nF(M) \to \varprojlim F(M/\mathfrak{a}^nM)$ is injective.

Proof. By 6.1.11 the module $\bigoplus_{n\geq 0} F(\mathfrak{a}^nM)$ is naturally a finite graded module over the Rees ring $R := \bigoplus_{n\geq 0} \mathfrak{a}^nT^n \subseteq A[T]$. Therefore we can find an integer $k \geq 0$ such that

$$F(\mathfrak{a}^{n+k}M) = \mathfrak{a}^nF(\mathfrak{a}^kM) \qquad \text{for all } n \in \mathbb{N}.$$

From the exact sequence

$$\mathfrak{a}^n F(\mathfrak{a}^k M) = F(\mathfrak{a}^{n+k} M) \longrightarrow F(M) \longrightarrow F(M/\mathfrak{a}^{n+k} M)$$

we get an exact sequence of inverse systems

$$\left\{\mathfrak{a}^n F(\mathfrak{a}^k M)/\mathfrak{a}^{n+k} F(M)\right\} \to \left\{F(M)/\mathfrak{a}^{n+k} F(M)\right\} \to \left\{F(M/\mathfrak{a}^{n+k} M)\right\} .$$

Since the system on the left is essentially zero, the result follows. ☐

Cohomology functors and the generic principle. Let X be a complex space and assume that for every open subset $U \subseteq X$ we have functors

$$F^i : \mathbf{Coh}(U) \longrightarrow \mathbf{Coh}(U) \ , \quad i \in \mathbb{Z}$$

satisfying the following conditions.

(a) F^i is $\Gamma(U, \mathcal{O}_U)$–linear i.e. for coherent sheaves $\mathcal{M}, \mathcal{N}$ on U the map

$$F_i : \operatorname{Hom}_U(\mathcal{M}, \mathcal{N}) \longrightarrow \operatorname{Hom}_U\left(F^i(\mathcal{M}), F^i(\mathcal{N})\right)$$

is $\Gamma(U, \mathcal{O}_U)$–linear.

(b) F^i is compatible with restrictions, that is, for $\mathcal{M} \in \mathbf{Coh}(U)$ and an open subset $V \subseteq U$ there are natural isomorphisms $F^i(\mathcal{M}) \mid V \cong F^i(\mathcal{M} \mid V)$.

(c) $\left\{F^i\right\}_{i\in\mathbb{Z}}$ constitutes a system of cohomology functors, i.e. for every sequence of coherent modules on some open subset $U \subseteq X$

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

there is a long exact sequence

$$\cdots \xrightarrow{\partial^{i-1}} F^i(\mathcal{M}') \longrightarrow F^i(\mathcal{M}) \longrightarrow F^i(\mathcal{M}'') \xrightarrow{\partial^i} F^{i+1}(\mathcal{M}') \longrightarrow \cdots$$

where the ∂^i are maps that are functorial with respect to short exact sequences in the usual sense.

It follows in particular that for coherent $\mathcal{O}_U$–modules $\mathcal{M}$, $\mathcal{N}$ we get an $\mathcal{O}_U$–linear map

$$\mathcal{H}om_U(\mathcal{M}, \mathcal{N}) \longrightarrow \mathcal{H}om_U\left(F^i(\mathcal{M}), F^i(\mathcal{N})\right) \ .$$

In the special case $\mathcal{M} = \mathcal{O}_U$ this amounts to a natural homomorphism

$$\mathcal{N} \otimes_{\mathcal{O}_U} F^i(\mathcal{O}_U) \longrightarrow F^i(\mathcal{N}) \ .$$

With these notations we have the following result.

THEOREM 6.1.13. *Let $X' \subseteq X$ be a reduced subspace of X and let $k \in \mathbb{Z}$ be fixed. Then there is a Zariski open dense subset $U \subseteq X'$ such that for every $V \subseteq U$ and every $\mathcal{O}_V$–module $\mathcal{N}$ the homomorphism*

$$\mathcal{N} \otimes_{\mathcal{O}_V} F^k(\mathcal{O}_V) \longrightarrow F^k(\mathcal{N})$$

is bijective. In particular, F^k satisfies the generic principle.

PROOF. We may suppose that $X' = X$. Clearly, if the result holds for every irreducible component of X, then it also holds for X. Therefore we can assume that X is irreducible of dimension, say, n. Let $U \subseteq X$ be the subset of points x such that

(a) $x \in \operatorname{Reg} X$,

(b) $F^{k+\nu}(\mathcal{O}_X)_x$ is a free $\mathcal{O}_{X,x}$–module for $\nu = 0, \ldots, n$.

Clearly U is Zariski open and dense in X. We claim that U is a set as required. Let $V \subseteq U$ be open and $\mathcal{N}$ a coherent $\mathcal{O}_V$–module. We need to show that the map $(*)$ is an isomorphism. It is sufficient to prove this locally in V. Therefore we may assume that there is a finite resolution

$$0 \longrightarrow \mathcal{F}_n \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{N} \longrightarrow 0 \ ,$$

where $\mathcal{F}_0, \ldots, \mathcal{F}_n$ are free coherent $\mathcal{O}_V$–modules. Consider $\mathcal{N}_r := \operatorname{coker}(\mathcal{F}_{r+1} \to \mathcal{F}_r)$ so that $\mathcal{N}_0 = \mathcal{N}$ and $\mathcal{N}_n = \mathcal{F}_n$. We will show by descending induction on r that the natural maps

$$(*)_r \qquad \mathcal{N}_r \otimes F^{k+\nu}(\mathcal{O}_V) \xrightarrow{\sim} F^{k+\nu}(\mathcal{N}_r) \ , \quad 0 \le \nu \le r,$$

are isomorphisms. If $r = n$ then $\mathcal{N}_n = \mathcal{F}_n$ is free, and the claim follows from the fact that each F^i compatible with finite direct sums.

Assume now that $(*)_{r+1}$ $(r \ge 0)$ is already shown and consider the exact sequence

$$0 \longrightarrow \mathcal{N}_{r+1} \longrightarrow \mathcal{F}_{r+1} \longrightarrow \mathcal{N}_r \longrightarrow 0 \ .$$

For each ν, we get a natural diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F^{k+\nu}(\mathcal{N}_{r+1}) \arrow[r, "j_\nu"] & F^{k+\nu}(\mathcal{F}_{r+1}) \arrow[r, "p_\nu"] & F^{k+\nu}(\mathcal{N}_r) \arrow[r] & \cdots \\
0 \arrow[r] & \mathcal{N}_{r+1} \otimes F^{k+\nu}(\mathcal{O}_V) \arrow[r] \arrow[u, "\alpha_\nu"] & \mathcal{F}_{r+1} \otimes F^{k+\nu}(\mathcal{O}_V) \arrow[r] \arrow[u, "\cong"] & \mathcal{N}_r \otimes F^{k+\nu}(\mathcal{O}_V) \arrow[r] \arrow[u, "\beta_\nu"'] & 0 \ .
\end{tikzcd}$$

Note that the bottom row is exact for $\nu = 0, \ldots, n$, by our assumption (b). By induction hypothesis the maps α_ν, $\nu = 0, \ldots, r+1$, are bijective. In particular it follows that j_ν is injective for $\nu = 0, \ldots, r+1$ and that p_ν is surjective for $\nu = 0, \ldots, r$. Now a simple diagram chase shows that β_ν is an isomorphism for $\nu = 0, \ldots, r$. ☐

6.2. Applications

In this section we will derive from our general criterion in Section 5.1 that for deformations of manifolds and deformations of vector bundles we always have openess of formal versality. The crucial step in proving this is to establish the existence of an obstruction theory satisfying the requirements of 6.1.10. We do this first for deformations of compact complex manifolds and, even more generally, for arbitrary locally trivial deformations of compact complex spaces. We derive that for such deformations openess of versality holds. In the second part we establish the existence of an obstruction theory for deformations of vector bundles. Again openess of versality follows.

Locally trivial deformations of complex spaces. Our first result is the main tool for constructing an obstruction theory. We consider the following notations.

Let $X \to S$ be a fixed flat morphism of compact complex spaces which is locally trivial. Moreover, $\mathcal{M}$ denotes a coherent $\mathcal{O}_S$-module.

THEOREM 6.2.1. *There exists a natural map*

$$ob : \operatorname{Ex}(S, \mathcal{M}) \longrightarrow H^2(X, \Theta_{X/S} \otimes f^*\mathcal{M})$$

such that for an extension $S \hookrightarrow S'$ *of* S *by* $\mathcal{M}$ *the class* $\mathrm{ob}([S'])$ *vanishes if and only if there exits a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
S \arrow[r, hook] & S',
\end{tikzcd}$$

where X' *is a locally trivial extension of* X *by* $f^*\mathcal{M}$*.*

For the proof we need the following lemma.

Lemma 6.2.2. *With the assumptions as in the proposition, suppose that* X *is a Stein space and* X' *and* $\tilde{X}'$ *are two* S'*-extensions of* X *by* $f^*(\mathcal{M})$*. Then there is an* S'*-isomorphism* $X' \overset{\varphi}{\cong} \tilde{X}'$ *of extension, i.e.* φ *induces the identity on* X*.*

Proof. Let $p : \mathbf{E} \to \mathbf{An}$ be the fibration in groupoids of locally trivial deformations, i.e. we consider all holomorphic maps $Z \to T$ that are flat and locally trivial. Then $a := (X \longrightarrow S)$ may be considerd as an object in $\mathbf{E}$, and $a' = (X' \to S')$ and $\tilde{a}' = (\tilde{X}' \to S')$ are extensions of a by $\mathcal{M}$. In the Kodaira-Spencer sequence

$$\mathrm{Ex}(a/S, \mathcal{M}) \longrightarrow \mathrm{Ex}(a, \mathcal{M}) \xrightarrow{p} \mathrm{Ex}(S, \mathcal{M})$$

the classes of $a', \tilde{a}'$ in $\mathrm{Ex}(a, \mathcal{M})$ map both onto $[S']$ in $\mathrm{Ex}(S, \mathcal{M})$. By 3.3.10

$$\mathrm{Ex}(a/S, \mathcal{M}) \cong H^1(X, \Theta_{X/S} \otimes f^*(\mathcal{M})),$$

and X being Stein this group vanishes. Hence p is injective and $a' \cong \tilde{a}'$. □

Proof of the theorem. Since the map $X \to S$ is locally trivial we can find an open covering $\mathcal{U} := \{X_i\}_{i \in I}$ of X by Stein open sets together with open embeddings

$$\varphi_i : X_i \hookrightarrow Y_i \times S_i.$$

Set $\mathcal{O}_{X'_i} := \varphi_i^*(\mathcal{O}_{Y_i \times S'_i})$ so that $X_i \hookrightarrow X'_i$ is an S' extension of X_i by $f^*(\mathcal{M})|X_i$ and $X'_i \to S'$ is locally trival. Consider $X_{ij} := X_i \cap X_j$ and

$$X_{ij} \hookrightarrow X'_i|X_{ij}, \qquad X_{ij} \hookrightarrow X'_i|X_{ij}.$$

These are two S' extensions of X_{ij} by $f^*(\mathcal{M})|X_{ij}$. By the preceding lemma they are isomorphic, i.e. there are S'-isomorphisms $\varphi_{ji} : X'_i|X_{ij} \to X'_j|X_{ij}$ with $\varphi_{ij} = \varphi_{ji}^{-1}$ restricting to the identity on X_{ij}. The map

$$\varphi_{ijk} := \varphi_{jk} \circ \varphi_{ki} \circ \varphi_{ij} : X'_j|X_{ijk} \longrightarrow X'_j|X_{ijk}$$

is an S' automorphism, where $X_{ijk} := X_i \cap X_j \cap X_k$. Thus it has the form $1 - \varepsilon\vartheta_{ijk}$ with an S-derivation

$$\vartheta_{ijk} : \mathcal{O}_{X_{ijk}} \longrightarrow f^*(\mathcal{M})|X_{ijk}.$$

The relation

$$id = \varphi_{ikl}^{-1}\varphi_{jkl}(\varphi_{kj}\,\varphi_{ijl}\varphi_{jk})(\varphi_{kj}\,\varphi_{ijk}^{-1}\,\varphi_{jk}).$$

show that

$$\vartheta_{jkl} - \vartheta_{ikl} + \vartheta_{ijl} - \vartheta_{ijk} = 0,$$

i.e. (ϑ_{ijk}) defines a Čech cocycle. We defines $\mathrm{ob}([S'])$ to be its Čech cohomology class in

$$H^2(\mathcal{U}, \Theta_{X/S} \otimes f^*\mathcal{M}) \cong H^2(X, \Theta_{X/S} \otimes f^*\mathcal{M}).$$

Let us prove that $\mathrm{ob}([S'])$ is well defined. First we show that it does not depend on the choice of the isomorphisms φ_{ij}. In fact, for another choice $\tilde{\varphi}_{ij}$ we have

$$\varphi_{jk} = \tilde{\varphi}_{ij} + \varepsilon\vartheta_{ij}$$

for some S'-derivations

$$\vartheta_{ij} : \mathcal{O}_{X_{ij}} \longrightarrow f^*(\mathcal{M})|X_{ij}.$$

Then the reader may verify that for the cocycles $(\vartheta_{ijk}), (\tilde{\vartheta}_{ijk})$ corresponding to $(\varphi_{ij}), (\tilde{\varphi}_{ijk})$, respectively, we have

$$(\vartheta_{ijk}) = (\tilde{\vartheta}_{ijk}) + \delta(\delta_{ij}),$$

where δ is the boundary operator in the Čech complex. Thus (ϑ_{ijk}) and $(\tilde{\vartheta}_{ijk})$ define the same cohomolgy class.

Secondly, let us show that $\mathrm{ob}([S'])$ is independent of the choice of the trivializations φ_i. In fact, given two trivialization $\varphi_i, \tilde{\varphi}_i$ as above we get correspondingly two S' extensions $X'_i, \tilde{X}'_i$ of X_i by $f^*(\mathcal{M})|X_i$. By the lemma above, there is an S'-isomorhism $h_i : X'_i \xrightarrow{\sim} \tilde{X}'_i$ inducing the identity on X_i. Taking as $\tilde{\varphi}_{ij} = h_i\varphi_{ij}h_i^{-1}$ we arrive at the same cocycle (ϑ_{ijk}). Finally, the independence from the choice of the covering is clear since any two coverings admit a common refinement.

Assume that $\mathrm{ob}([S'])$ vanishes. Then we can write

$$\vartheta_{ijk} = \vartheta_{jk} - \vartheta_{ik} + \vartheta_{ij}$$

for a collection of S-derivations

$$\vartheta_{ij} : \mathcal{O}_{X_{ij}} \to f^*(\mathcal{M})|X_{ij} \quad \text{with} \quad \vartheta_{ij} = -\vartheta_{ji}.$$

Then $h_{ij} := 1 - \varepsilon\vartheta_{ij}$ defines an automorphism of $X'_i|X_{ij}$. Replacing φ_{ij} by

$$\psi_{ij} := \varphi_{ij} \circ h_{ij}^{-1} = \varphi_{ij} + \varepsilon\vartheta_{ij}$$

we obtain $\psi_{ij}|X_{ij} = \mathrm{id}_{X_{ij}}$ and

$$\psi_{jk}\,\psi_{ik}^{-1}\,\psi_{ij} = \mathrm{id}\,.$$

Hence, pasting the X'_i along the isomorphisms ψ_{ij} we obtain an S'-space X' which is an extension of X by $f^*(\mathcal{M})$, and by construction $X' \longrightarrow S'$ is locally trivial.

Conversely, assume that there is an S'-extension $X \hookrightarrow X'$ by $f^*(\mathcal{M})$ such that $X' \to S'$ is locally trivial. Choose the covering $\mathcal{U} = (X_i)_{i\in I}$ in such a way that there are already trivializations $X'_i \hookrightarrow X_{i0} \times S'$. Restricting them to X_i gives trivializations φ_i as above. Now the construction shows immediatily that the corresponding cocycle (ϑ_{ijk}) vanishes and so $\mathrm{ob}([S]) = 0$.

Finally, it is immediate from the construction that the map ob is functorial in $\mathcal{M}$ and compatible with restrictions to open sets. This completes the proof of the theorem. □

Let now $p : \mathbf{E} \to \mathbf{An}$ denot the groupoid of lacally trivial deformations so that our given map $X \to S$ map be considered as an object of $\mathbf{E}$ which we denote by a. Sheafifying the map ob constructed in 6.2.1 we get a functorial map also denoted by ob

$$(*) \qquad \mathrm{ob} : \mathcal{E}x(S, \mathcal{M}) \longrightarrow R^2 f_*(\Theta_{X/S} \otimes f^*\mathcal{M}).$$

The above result 6.2.1 amounts to the exactness of the sequence

$$\mathrm{Ex}(a, \mathcal{M}) \longrightarrow \mathrm{Ex}(S, \mathcal{M}) \xrightarrow{ob} H^2(S, \Theta_{X/S} \otimes f^*\mathcal{M}).$$

Hence we obtain the following result.

COROLLARY 6.2.3. *Assume that $X \to S$ is a proper map which is locally trivial. Then the map* ob *in* $(*)$ *is an obstruction theory for* $a = (X \to S)$.

PROOF. Shefifying the above sequence we get an exact sequence

$$\mathcal{E}x(a,\mathcal{M}) \longrightarrow \mathcal{E}x(S,\mathcal{M}) \xrightarrow{ob} R^2 f_*(\Theta_{X/S} \otimes f^*\mathcal{M}).$$

By the properness of f the sheaf on the right hand side is coherent. This proves the result. □

Applying our criterion for openess of versality 6.1.10 gives the following result.

THEOREM 6.2.4. *Let $f : X \to S$ be a proper map of complex spaces which is locally trivial. Then openess of versality holds for f, i.e. the set of points $s \in S$, where f is a versal deformation, is Zariski open in S.*

PROOF. It remains to show that the conditions (O1), (O2) in 6.1.10 are satisfied. First, by **??**

$$\mathrm{Ex}(a/S,\mathcal{M}) \cong R' f_*(\Theta_{X/S} \otimes f^*\mathcal{M})$$

is coherent on S, i.e. (O1) is satisfied. Applying **??** to the cohomolgy functors $F^i(-) := R^i f_*(\Theta_{X/S} \otimes f^*(-))$ we obtain that (O2) also holds. The result follows. □

As a special case we note the case of deformations of compact complex manifolds.

COROLLARY 6.2.5. *Let $f : X \to S$ be a proper smooth map. Then openess of versality holds for f.*

Deformations of modules. Let us now turn to deformations of modules. For the remaining part of this section we use the following notations.

Let $X \to \Sigma$ be a fixed flat holomorphic map. For a Σ-space S set $\mathcal{X} := X \times_\Sigma S$ so that $f : \mathcal{X} \longrightarrow S$ is flat. In our first result we consider extensions of a fixed locally free sheaf $\mathcal{F}$ of finite rank on $\mathcal{X}$.

THEOREM 6.2.6. *(1) There is a natural map*

$$\mathrm{ob} = \mathrm{ob}_\mathcal{F} : \mathrm{Ex}_\Sigma(S,\mathcal{M}) \longrightarrow H^2(\mathcal{X}, \mathcal{E}nd\,(\mathcal{F}) \otimes f^*\mathcal{M})$$

such that for an Σ-extension $(S \hookrightarrow S')$ of S by $\mathcal{M}$ the class $\mathrm{ob}([S'])$ *vanishes if and only if there is a locally free sheaf $\mathcal{F}'$ on $\mathcal{X}' := X \times_\Sigma S'$ restricting to $\mathcal{F}$ on $\mathcal{X} \hookrightarrow \mathcal{X}'$.*

(2) Assumes that X is compact, S is Stein and Σ is a simple point. Then $\mathrm{ob}([S'])$ *vanishes under the trace map*

$$tr : H^2(\mathcal{X}, \mathcal{E}nd\,(\mathcal{F} \otimes f^*\mathcal{M}) \longrightarrow H^2(\mathcal{X}, f^*\mathcal{M}),$$

and so $\mathrm{ob}([S']) \in H^2(\mathcal{X}, sl(\mathcal{F}) \otimes f^*\mathcal{M})$.

The proof is similar to the proof of 6.2.1. First we note the following analogue of 6.2.2.

LEMMA 6.2.7. *In the situation of 6.2.6, assume that X is a Stein space and that $\mathcal{F}'$ and $\tilde{\mathcal{F}}'$ are two locally free sheaves on X' restricting to $\mathcal{F}$. Then there is an isomorphism $\mathcal{F}' \cong \tilde{\mathcal{F}}'$ inducing the identity on $\mathcal{F}$.*

PROOF. Let $p : \mathbf{E} \to \mathbf{An}_\Sigma$ be the deformation theory, where the objects of $\mathbf{E}$ are given by pairs $(\mathcal{E}, T)$, with $T \in \mathbf{An}_\Sigma$ and $\mathcal{E}$ a vector bundle on $X \times_\Sigma T$. Similarly as in the proof of 6.2.1 we can consider the classes of $a' = (\mathcal{F}', S'), \tilde{a}' = (\tilde{\mathcal{F}}', S')$ in $\mathrm{Ex}_\Sigma(a, \mathcal{M})$, where $a \in \mathbf{E}$ is the object given by $(\mathcal{F}, S)$. Again, it is sufficient to verify that $\mathrm{Ex}_\Sigma(a/S, \mathcal{M})$ vanishes. But this follows from the fact that X is Stein and so by 5.3.2

$$\mathrm{Ex}_\Sigma(a/S, \mathcal{M}) \cong H^1(X, \mathcal{E}nd\,(\mathcal{E}) \otimes f_S^* \mathcal{M}) = 0$$

□

PROOF OF 6.2.6. Let $\mathcal{U} = (\mathcal{X}_i)_{i \in I}$ be a covering of $\mathcal{X}$ such that $\mathcal{F}|\mathcal{X}_i \cong \mathcal{O}^r_{\mathcal{X}_i}$. Let $\mathcal{X}_{i_0 \ldots i_k}$ denote the intersection $\mathcal{X}_{i_0} \cap \ldots \cap \mathcal{X}_{i_k}$, and similary for $\mathcal{X}'_{i_0 \ldots i_k}$. We set $\mathcal{F}'_i := \mathcal{O}^r_{\mathcal{X}'_i}$. By the lemma above, there are isomorphisms

$$\varphi_{ji} : \mathcal{F}'_i/\mathcal{X}'_{ij} \xrightarrow{\sim} \mathcal{F}'_j|\mathcal{X}'_{ij} \quad \text{with} \quad \varphi_{ji} = \varphi_{ij}^{-1}$$

inducing the identity on $\mathcal{F}|\mathcal{X}_{ij}$. We consider isomorphisms

$$\varphi_{ijk} := \varphi_{jk}\varphi_{ki}\varphi_{ij} : \mathcal{F}'_j|\mathcal{X}'_{ijk} \longrightarrow \mathcal{F}'_j|\mathcal{X}'_{ijk}.$$

When tensored with $\mathcal{O}_{\mathcal{X}_{ijk}}$ they are the identity on $\mathcal{F}|\mathcal{X}_{ijk}$. Hence we can write $\varphi_{ijk} = 1 - \varepsilon\vartheta_{ijk}$, where

$$\vartheta_{ijk} \in H^\circ(\mathcal{X}_{ijk}, \mathcal{E}nd\,(\mathcal{E}) \otimes f^*\mathcal{M}).$$

The same calculation as in the proof of 6.2.2 shows that (ϑ_{ijk}) is a Čech cocycle and so defines a cohomology class

$$\mathrm{ob}_\mathcal{F}([S']) \in \check{H}^2(\mathcal{X}, \mathcal{E}nd\,(\mathcal{E}) \otimes f^*\mathcal{M}) \cong H^2(\mathcal{X}, \mathcal{E}nd\,(\mathcal{E}) \otimes f^*\mathcal{M}).$$

Moreover by the same reasoning as in loc.cit. $\mathrm{ob}_\mathcal{F}([S'])$ is independent of the choices involved and the construction is functorial with respect to $\mathcal{M}$.

Assume that $\mathrm{ob}_\mathcal{F}([S'])$ vanishes so that we can write

$$\vartheta_{ijk} = \vartheta_{jk} - \vartheta_{ik} + \vartheta_{ij},$$

where $\vartheta_{ij} = -\vartheta_{ij}$ are in $H^0(\mathcal{X}'_{ij}, \mathcal{E}nd\,(\mathcal{E}) \otimes f^*\mathcal{M})$. Then $h_{ij} := 1 - \varepsilon\vartheta_{ij}$ defines an automorphism of $\mathcal{F}'_i|\mathcal{X}'_{ij}$. Replacing φ_{ij} by $\psi_{ij} := \varphi_{ij} \circ h_{ij}^{-1}$ we have $\psi_{ki}\psi_{ij}\psi_{jk} = \mathrm{id}$, so that we can paste the bundles $\mathcal{F}'_i$ to obtain a vector bundle $\mathcal{F}'$ on $\mathcal{X}'$ restricting to $\mathcal{F}$ over S. Converseley, it follows with almost the same arguments as in the proof of 6.2.2 that the existence of $\mathcal{F}'$ implies the vanishing of $\mathrm{ob}([S'])$.

For the proof of (2) note first that

$$(a) \qquad H^2(\mathcal{X}, f^*\mathcal{M}) \cong H^0(S, R^2 f_*(\mathcal{O}_\mathcal{X}) \otimes \mathcal{M}),$$

since S is Stein. We need to show that $\mathrm{tr}(\mathrm{ob}_\mathcal{F}([S']))$ vanishes in this group. In a first step let us prove that

$$(b) \qquad \mathrm{tr}(\mathrm{ob}_\mathcal{F}([S']) = \mathrm{ob}_\mathcal{L}([S']),$$

where $\mathcal{L} := \det \mathcal{F}$ is the determinant bundle of $\mathcal{F}$; note that $\mathcal{E}nd\,\mathcal{L} \cong \mathcal{O}_\mathcal{X}$ and so $\mathrm{ob}_\mathcal{L}([S'])$ is an element of the group in (a). Consider in the construction above $\mathcal{L}'_i := \det \mathcal{F}'_i$ instead of $\mathcal{F}'_i$, and set $\Phi_{ij} := \det \varphi_{ij}$, $\Phi_{ijk} := \det \varphi_{ijk}$, so that $\Phi_{ijk} = \Phi_{jk} \circ \Phi_{ki} \circ \Phi_{ij}$. As $\varphi_{ijk} = 1 - \varepsilon\vartheta_{ijk}$ and $\varepsilon^2 = 0$ we have

$$\Phi_{ijk} = 1 - \varepsilon\,\mathrm{tr}(\vartheta_{ijk}).$$

Hence by the construction above $\mathrm{tr}(\vartheta_{ijk})$ represents $\mathrm{ob}_\mathcal{L}([S'])$ which shows the equality in (b).

In a second step let us verify that $\mathrm{ob}_{\mathcal{L}}([S'])$ vanishes. Because of (1) it suffices to show this locally around each point $s \in S$. By the universal property of the Picard scheme, the bundle $\mathcal{L}$ amounts to a map $\alpha : (S, s) \to \operatorname{Pic} X$ into the Picard variety. As $\operatorname{Pic} X$ is smooth we can lift α to a map $(S', s) \to \operatorname{Pic} X$. Then the pullback of the universal bundle is an extension $\mathcal{L}'$ of $\mathcal{L}$ in a neighbourhood of s. Thus $\mathrm{ob}_{\mathcal{L}}([S'])$ vanishes as a section in $H^0(S, R^2 f_* \mathcal{O}_{\mathcal{X}})$ as required. □

Like in the case of locally trivial deformations of spaces, we use this result to establish the existence of an obstruction theory in the sense of 6.1.6. As before let $p : \mathbf{E} \to \mathbf{An}_\Sigma$ denote the deformation theory of vector bundles on X, i.e. the objects of $\mathbf{E}$ are given by pairs $(\mathcal{S}, \mathcal{F})$, where $S \in \mathbf{An}_\Sigma$ and $\mathcal{F}$ is a locally free coherent sheaf an $\mathcal{X} = X \times_\Sigma S$. The above result amounts to the exactness of the sequence

$$\mathcal{E}x_\Sigma(a, \mathcal{M}) \longrightarrow \mathcal{E}x_\Sigma(S, \mathcal{M}) \xrightarrow{\mathrm{ob}} R^2 f_*(\mathcal{E}nd\,(\mathcal{E}) \otimes f^* \mathcal{M}),$$

where ob is the sheafified map ob from 6.2.7. Hence we obtain the following corollary.

Corollary 6.2.8. *Let $X \to \Sigma$ be as above and assume moreover that this map is proper. Then the map* ob *in the above sequence is an obstruction theory for $(\mathcal{F}, S)$.*

Proof. It remains to verify that $R^2 f_*(\mathcal{E}nd\,(\mathcal{E}) \otimes f^* \mathcal{M})$ is coherent. But this is clear from our assumption. □

Using the same line of arguments as in 6.2.4 we derive the following result.

Theorem 6.2.9. *Let $X \to \Sigma$ be as in 6.2.7 and $\mathcal{F}$ a vector bundle on $\mathcal{X} = X \times_\Sigma S$. Then the set of points $s \in S$ in which $\mathcal{F}$ is the versal deformation of its fibre, is Zariski open in S.*

Remark 6.2.10. More generally one has an obstruction theory for an arbitrary coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ which has proper support over S and is S-flat. In this case the obstructions lie in $\operatorname{Ext}^2(\mathcal{F}, \mathcal{F} \otimes f^* \mathcal{M})$, and one can again derive openess of versality, see **??** for details.

6.3. Universal deformations

Let $p : \mathbf{F} \to \mathbf{An}$ be a deformation theory, $S \in \mathbf{An}$ a complex space and $a \in \mathbf{F}(S)$. Let us first give a simple criterion for when an extension $a \hookrightarrow a'$ is trivial, see also 2.3.3.

Lemma 6.3.1. *If $a \hookrightarrow a'$ is an extension of a by a coherent $\mathcal{O}_S$-module $\mathcal{M}$, then the following are equivalent.*

1. *There is a section $\sigma : a' \to a$ so that $a \to a' \to a$ is the identity.*
2. *$a' \cong a[\mathcal{M}]$ is the trivial extension of a by $\mathcal{M}$.*

Moreover, the isomorphism in (2) is uniquely determined by the section σ.

Proof. This follows easily from the universal property (**FC1**) in 3.1.1. □

LEMMA 6.3.2. *Let* $a \hookrightarrow a'$ *be an extension of* a *by* $\mathcal{M}$*. Then there is a canonical isomorphism* $a' \coprod_a a' \cong a' \coprod_a a[\mathcal{M}]$ *such that the diagram*

$$\begin{tikzcd}
& a' \arrow[dl, hook', "i"'] \arrow[dr, hook, "j"] & \\
a'\coprod_a a' \arrow[rr, equal, "\sim"] & & a'\coprod_a a[\mathcal{M}]
\end{tikzcd}$$

commutes, where i*,* j *are the inclusions into the first summand.*

PROOF. The map

$$\mathrm{id} \coprod \mathrm{id} : a' \coprod_a a' \longrightarrow a'$$

is a section of $a \hookrightarrow a'$, whence by 6.3.1 $a' \coprod_a a' \cong a'[\mathcal{M}]$. As the latter object is isomorphic to $a' \coprod_a a[\mathcal{M}]$, the result follows. ☐

COROLLARY 6.3.3. $\mathrm{Aut}_a\, a' \cong \mathrm{Aut}_a\, a[\mathcal{M}]$.

PROOF. By 3.2.4 we have

$$\mathrm{Aut}_a\,(a' \textstyle\coprod_a a') \cong \mathrm{Aut}_a\, a' \times \mathrm{Aut}_a\, a'$$
$$\mathrm{Aut}_a\,(a' \textstyle\coprod_a a[\mathcal{M}]) \cong \mathrm{Aut}_a\, a' \times \mathrm{Aut}_a\, a[\mathcal{M}].$$

Hence the diagram in 6.3.2 yields a commutative diagram

$$\begin{tikzcd}
& \mathrm{Aut}_a\, a' & \\
\mathrm{Aut}_a\, a' \times \mathrm{Aut}_a\, a' \arrow[ur, "p"] \arrow[rr, equal, "\sim"] & & \mathrm{Aut}_a\, a' \times \mathrm{Aut}_a\, a[\mathcal{M}]\,, \arrow[ul, "q"']
\end{tikzcd}$$

where p, q denote the projections onto the first factor. Thus

$$\mathrm{Aut}_a\, a' \cong p^{-1}(\mathrm{id}_{a'}) \cong q^{-1}(\mathrm{id}_{a'}) \cong \mathrm{Aut}_a\, a[\mathcal{M}]\,,$$

as required. ☐

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